A Note on Additional Materials for "Feedback and Contagion through Distressed Competition"

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Abstract

This note contains a continuous-time version of the full-fledged quantitative model in the paper titled "Feedback and Contagion through Distressed Competition" (Chen et al., 2023*a*). Section 1 develops the model in continuous time and provides additional discussions on the literature. Section 2 clarifies some technical conditions for the boundary conditions of the continuous-time model. Section 3 provides the numerical algorithm that solves the continuous-time model using a discrete-time dynamic programming method. Moreover, an internet appendix (Chen et al., 2023*b*) provides empirical evidence supporting the theoretical results, as well as additional analyses of the full-fledged quantitative model.

1 Additional Materials on the Quantitative Model

We develop a continuous-time version of the full-fledged quantitative model. Section 1.1 presents the setup of the model. Section 1.2 formulates the Nash equilibrium. Section 1.3 provides additional discussions on the literature.

1.1 Model Formulation

The model describes an infinite-horizon economy with continuous time $t \geq 0$ and two firms within the same industry. The continuously compounded risk-free rate is $r_f \geq 0$. At $t = 0$, firms are financed by external equity and long-term consol debt at coupon rate e^{b_i}. Firms engage in Bertrand competition over every instant of time.

Product Market Competition and Cash Flows. Firm i's earnings intensity after interest expenses over $[t, t + \mathrm{d}t)$ is

$$\mathcal{E}_{i,t} = (P_{i,t} - \omega)e^{z_{i,t}}C_{i,t} - e^{b_i}, \tag{1}$$

where ω is the marginal cost of production, $P_{i,t}$ is the price, and $e^{z_{i,t}}C_{i,t}$ is the demand, which is determined by a firm-specific demand intensity $z_{i,t}$, for firm $i \in \{1, 2\}$. The timeline of a firm's decisions and actions within every instant of time can be summarized as follows. A firm first observes its demand intensity, then names its price and receives the demand for its goods. Next, the firm uses its operating cash flows to make interest payments, and then decides whether to default. Finally, if it chooses not to default, the firm pays taxes and distributes the remaining cash flows to its shareholders as dividends. Specifically, firm i's earnings intensity after interest expenses and taxes is $(1 - \tau)\mathcal{E}_{i,t}$, where the corporate tax rate is $\tau \in (0, 1)$.

Within every instant of time $[t, t + \mathrm{d}t)$, firms simultaneously name their prices $P_{1,t}$ and $P_{2,t}$, knowing that they face downward-sloping within- and cross-industry demand functions. Once prices are set, the demand $e^{z_{1,t}}C_{1,t}$ and $e^{z_{2,t}}C_{2,t}$ are determined according to the following demand system. To characterize how industry demand depends on the industry-level price index P_t, we follow the literature (e.g., Hopenhayn, 1992; Pindyck, 1993; Caballero and Pindyck, 1996) and postulate a downward-sloping isoelastic industry demand curve. Specifically, the industry demand is $e^{a_t}C_t$, where $e^{a_t} \equiv \sum_{i=1}^{2} e^{z_{i,t}}$ is the industry-level demand intensity and C_t is given by

$$C_t = P_t^{-\epsilon}, \tag{2}$$

where the parameter $\epsilon > 1$ captures the industry-level price elasticity of demand.

Next, we introduce the demand system for differentiated goods within an industry. Given C_t and P_t, consumers decide on a basket of differentiated goods, which are produced by the two firms, based on the prices $P_{1,t}$ and $P_{2,t}$ charged by firms 1 and 2, respectively. Specifically, C_t equals a Dixit-Stiglitz constant elasticity of substitution (CES) aggregator:

$$C_t = \left[\sum_{i=1}^{2} \left(e^{z_{i,t} - a_t} \right) C_{i,t}^{\frac{\eta - 1}{\eta}} \right]^{\frac{\eta}{\eta - 1}}, \tag{3}$$

where the parameter $\eta > 1$ captures the elasticity of substitution among goods produced by the two firms in the same industry. Intuitively, the weight $e^{z_{i,t} - a_t}$ captures consumers' relative preference for firm i's goods. We assume that $\eta \geq \epsilon > 1$, meaning that goods within the same industry are more substitutable than those across industries.

From the CES aggregator, the firm-level demand curve immediately follows. Specifically, given the prices $P_{i,t}$ for $i = 1, 2$ and C_t, the demand for firm i's goods $e^{z_{i,t}} C_{i,t}$ can be obtained by solving a standard expenditure minimization problem:

$$e^{z_{i,t}} C_{i,t} = e^{z_{i,t} - a_t} \left(\frac{P_{i,t}}{P_t} \right)^{-\eta} e^{a_t} C_t, \quad \text{with the price index } P_t = \left[\sum_{i=1}^{2} e^{z_{i,t} - a_t} P_{i,t}^{1-\eta} \right]^{\frac{1}{1-\eta}}. \tag{4}$$

All else equal, the demand for firm i's goods $e^{z_{i,t}} C_{i,t}$ increases with consumers' relative preference $e^{z_{i,t} - a_t}$ for firm i's goods in equilibrium. A larger $e^{z_{i,t} - a_t}$ implies that firm i's price $P_{i,t}$ has a greater influence on the price index P_t. The demand curves at the industry level (2) and the firm level (4) yield:

$$C_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\eta} P_t^{-\epsilon}. \tag{5}$$

The short-run price elasticity of demand for firm i's goods is

$$-\frac{\partial \ln(e^{z_{i,t}} C_{i,t})}{\partial \ln P_{i,t}} = \underbrace{\mu_{i,t} \left[-\frac{\partial \ln(e^{z_{i,t}} C_{i,t})}{\partial \ln P_t} \right]}_{\text{cross-industry}} + \underbrace{(1 - \mu_{i,t}) \left[-\frac{\partial \ln(e^{z_{i,t}} C_{i,t} / (e^{a_t} C_t))}{\partial \ln(P_{i,t} / P_t)} \right]}_{\text{within-industry}} = \mu_{i,t} \epsilon + (1 - \mu_{i,t}) \eta, \tag{6}$$

where $\mu_{i,t}$ is the (revenue) market share of firm i, which equals $\mu_{i,t} = e^{z_{i,t} - a_t} \left(P_{i,t} / P_t \right)^{1-\eta}$. Equation (6) shows that the short-run price elasticity of demand is given by the average of η and ϵ, weighted by the firm's market share $\mu_{i,t}$. When its market share $\mu_{i,t}$ shrinks (grows), within-industry (cross-industry) competition becomes more relevant for firm i, so its price elasticity of demand depends more on η (ϵ). There are two extreme cases, $\mu_{i,t} = 0$

and $\mu_{i,t} = 1$. In the former case, firm i becomes atomistic and takes the industry price index P_t as given; as a result, firm i's price elasticity of demand is exactly η. In the latter case, firm i monopolizes the industry, and its price elasticity of demand is exactly ϵ.

The firm-specific demand intensity $z_{i,t}$ follows a process with intertemporal dependence:

$$e^{-z_{i,t}}\mathrm{d}e^{z_{i,t}} = g\mathrm{d}t + \varsigma\mathrm{d}W_t + \sigma\mathrm{d}W_{i,t} - \mathrm{d}J_{i,t}, \tag{7}$$

where the parameter g captures the firm's expected growth rate, the standard Brownian motion W_t ($W_{i,t}$) captures aggregate (idiosyncratic) demand shocks, and the Poisson process $J_{i,t}$ with intensity ν captures idiosyncratic left-tail jump shocks in firm i's cash flows. Firm i exits the industry upon the occurrence of a Poisson shock. The shocks W_t, $W_{i,t}$, and $J_{i,t}$ are mutually independent. The coefficient ν captures idiosyncratic left-tail risk.

Several points regarding the shocks are worth mentioning. First, the aggregate Brownian shock W_t in equation (7) is an economy-wide or industry-wide demand shock. Second, the idiosyncratic Brownian shocks, $W_{1,t}$ and $W_{2,t}$, are firm-specific demand shocks. Idiosyncratic shocks are needed for the model to quantitatively match the default frequency and generate a nondegenerate cross-sectional distribution of market shares in the stationary equilibrium. Third, the idiosyncratic left-tail jump shocks, $J_{1,t}$ and $J_{2,t}$, play a crucial role in our theory and empirical results. Idiosyncratic left-tail risk has been proven useful in explaining credit spreads and credit default swap index (CDX) spreads (e.g., Delianedis and Geske, 2001; Collin-Dufresne, Goldstein and Yang, 2012; Kelly, Manzo and Palhares, 2018; Seo and Wachter, 2018).

In the rest of this section, we focus on illustrating firms' problems under the risk-neutral ($\mathbb{Q}$) measure. Specifically, let γ be the market price of risk for the aggregate shock W_t. Equation (7) can be written as:

$$e^{-z_{i,t}}\mathrm{d}e^{z_{i,t}} = (g - \varsigma\gamma)\mathrm{d}t + \varsigma\mathrm{d}W_t^{\mathbb{Q}} + \sigma\mathrm{d}W_{i,t} - \mathrm{d}J_{i,t}, \text{ with } \mathrm{d}W_t^{\mathbb{Q}} = \gamma\mathrm{d}t + \mathrm{d}W_t, \tag{8}$$

where $\mathrm{d}W_t^{\mathbb{Q}}$ captures the aggregate shocks under the risk-neutral measure.

Endogenous Profits and Externalities. Now, we characterize the profitability function. Firm i's operating profits are

$$(P_{i,t} - \omega)\, e^{z_{i,t}} C_{i,t} = \Pi_i(\theta_{i,t}, \theta_{j,t}) e^{z_{i,t}}, \text{ with } \Pi_i(\theta_{i,t}, \theta_{j,t}) \equiv \omega^{1-\epsilon}\theta_{i,t}\,(1-\theta_{i,t})^{\eta-1}\,(1-\theta_t)^{\epsilon-\eta}, \tag{9}$$

where $\theta_{i,t}$ and θ_t represent the firm-level and industry-level profit margins,

$$\theta_{i,t} \equiv \frac{P_{i,t} - \omega}{P_{i,t}} \text{ and } \theta_t \equiv \frac{P_t - \omega}{P_t}, \tag{10}$$

and it directly follows from equation (4) that the relation between $\theta_{i,t}$ and θ_t is

$$1 - \theta_t = \left[\sum_{i=1}^{2} e^{z_{i,t} - a_t} (1 - \theta_{i,t})^{\eta - 1} \right]^{\frac{1}{\eta - 1}}. \tag{11}$$

Equation (9) shows that firm i's profits depend on its rival j's profit margin $\theta_{j,t}$ through the industry's profit margin θ_t. This reflects the externality of firm j's profit margin decisions. For example, holding firm i's profit margin fixed, if firm j cuts its profit margin $\theta_{j,t}$, the industry's profit margin θ_t will drop, which will reduce the demand for firm i's goods $C_{i,t}$ (see equation (5)), compromising firm i's profits. Below, we explain the Nash equilibrium, which determines the profit margin strategies $(\theta_{1,t}, \theta_{2,t})$.

Financial Distress. Firm i can optimally choose to file for bankruptcy and exit when its equity value drops to zero because of negative shocks to the demand intensity $z_{i,t}$. To maintain tractability, a new firm enters the industry only after an incumbent firm exits. However, the new firm has an initial demand intensity $e^{z_{new}} = \kappa e^{z_{j,0}} > 0$ and an optimal coupon rate $e^{b_{new}}$. The parameter $\kappa > 0$ captures the size of the new entrant relative to the surviving incumbent firm j's initial size. Intuitively, a larger κ reflects a higher entry threat to the incumbent. Upon entry, the dynamic game of industry competition, which we describe in Section 1.2 below, is "reset" to a new one between the surviving incumbent firm and the new entrant

1.2 Nash Equilibrium

Non-Collusive Equilibrium. The two firms in an industry play a supergame (Friedman, 1971), in which the stage games of setting profit margins are continuously played and infinitely repeated with exogenous and endogenous state variables varying over time. All strategies depend on "payoff-relevant" states $z_t \equiv \{z_{1,t}, z_{2,t}\}$ in the state space $\mathcal{Z}$, as in Maskin and Tirole (1988*a*,*b*).

Specifically, the non-collusive equilibrium is characterized by a profit-margin-setting scheme $\Theta^N(\cdot) = (\theta_1^N(\cdot), \theta_2^N(\cdot))$, which is a pair of functions defined in the state space $\mathcal{Z}$, such that firm i's equity value $E_i^N(z_t)$ is maximized by choosing the profit margin $\theta_i^N(z_t)$, under the assumption that its rival j will stick to the non-collusive profit margin $\theta_j^N(z_t)$.

We denote by $\underline{z}_i^N(z_{j,t})$ firm i's endogenous default boundary with respect to $z_{i,t}$ in the non-collusive equilibrium; importantly, the endogenous default boundary of firm i depends on its rival's demand intensity $z_{j,t}$. Following the recursive formulation in dynamic games for characterizing the Nash equilibrium (e.g., Pakes and McGuire, 1994; Ericson and Pakes, 1995), we formulate the optimization problems, conditioning on $z_{i,t} > \underline{z}_i^N(z_{j,t})$ for $i \neq j \in \{1,2\}$, as a pair of Hamilton-Jacobi-Bellman (HJB) equations:

$$r_f E_i^N(z_t)\mathrm{d}t = \max_{\theta_{i,t}} \ (1-\tau)\left[\Pi_i(\theta_{i,t}, \theta_{j,t}^N)e^{z_{i,t}} - e^{b_i}\right]\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}E_i^N(z_t)], \tag{12}$$

where $i \neq j \in \{1,2\}$, $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ represents the expectation under the risk-neutral measure conditioning on the information set up to t, $\theta_{i,t}^N \equiv \theta_i^N(z_t)$, and $\Pi_i(\theta_{i,t}, \theta_{j,t})$ is defined in (9). The coupled HJB equations provide the solutions for the non-collusive profit margins $\theta_1^N(z_t)$ and $\theta_2^N(z_t)$.

At the default boundary $\underline{z}_i^N(z_{j,t})$ of firm i, the equity value of firm i is equal to zero (i.e., the value matching condition), and the optimality of the default boundary implies the smooth pasting condition:

$$E_i^N(z_t)\Big|_{z_{i,t}=\underline{z}_i^N(z_{j,t})} = 0 \quad \text{and} \quad \frac{\partial}{\partial z_{i,t}} E_i^N(z_t)\Big|_{z_{i,t}=\underline{z}_i^N(z_{j,t})} = 0, \quad \text{respectively.} \tag{13}$$

As $z_{i,t} \to +\infty$, firm i becomes an industry monopoly, which sets an asymptotic boundary condition (see Section 2.2 of this note). The value matching and smooth pasting conditions in (13) ensures that the smooth-pasting condition with respect to $z_{j,t}$ holds in equilibrium, i.e., $\frac{\partial}{\partial z_{j,t}} E_i^N(z_t)\Big|_{z_{i,t}=\underline{z}_i^N(z_{j,t})} = 0$. This property may not hold generally in other corporate liquidity models with multiple state variables and multiple boundaries (e.g., Chen et al., 2022; Kakhbod et al., 2022). We provide detailed proofs and discussions in Section 2.1 of this note.

Competition Under Tacit Collusion. Our main focus is the collusive equilibrium, which is sustained using the non-collusive equilibrium as a punishment strategy. Firms tacitly collude with each other in setting higher profit margins, with any deviation potentially triggering a switch to the non-collusive equilibrium.

In the collusive equilibrium, strategies not only depend on "payoff-relevant" states $z_t \equiv \{z_{1,t}, z_{2,t}\}$, but also on a pair of indicator functions that track whether either firm has previously deviated from the collusive agreement, as in Fershtman and Pakes (2000, p. 212).[1] Consider a generic collusive equilibrium in which the two firms follow a collusive

[1] For notational simplicity, we omit the indicator states of historical deviations.

profit-margin-setting scheme. If one firm deviates from the collusive profit-margin-setting scheme, then with a probability of $\xi \mathrm{d}t$ over $[t, t+\mathrm{d}t)$, the other firm will implement a punishment strategy, under which it will forever set the non-collusive profit margin. We use an idiosyncratic Poisson process $N_{i,t}$ with intensity ξ to characterize whether a firm can successfully implement a punishment strategy after the rival's deviation.[2] Thus, a higher ξ makes the threat of punishment more credible, which reduces incentives to deviate and enables collusion at higher profit margins.

Formally, the set of incentive compatible collusion agreements, denoted by $\mathcal{C}$, consists of all continuous profit-margin-setting schemes $\Theta^C(\cdot) \equiv (\theta_1^C(\cdot), \theta_2^C(\cdot))$ such that the following participation constraint (PC) and IC constraint are satisfied:

$$E_i^N(z) \leq E_i^C(z), \quad \text{for all } z \in \mathcal{Z}, \text{ and} \qquad \text{(PC)} \qquad (14)$$

$$E_i^D(z) \leq E_i^C(z), \quad \text{for all } z \in \mathcal{Z}, \qquad \text{(IC)} \qquad (15)$$

where $i \in \{1,2\}$, $E_i^D(z)$ is firm i's equity value if it chooses to deviate from collusion, and $E_i^C(z)$ is firm i's equity value in the collusive equilibrium.

We denote by $\underline{z}_i^C(z_{j,t})$ firm i's endogenous default boundary with respect to $z_{i,t}$ in the collusive equilibrium. Conditional on $z_{i,t} > \underline{z}_i^C(z_{j,t})$ for $i \neq j \in \{1,2\}$, the value functions $E_i^C(z)$ satisfy the following coupled HJB equations:

$$r_f E_i^C(z_t)\mathrm{d}t = (1-\tau)[\Pi_i(\theta_{i,t}^C, \theta_{j,t}^C)e^{z_{i,t}} - e^{b_i}]\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}E_i^C(z_t)], \qquad (16)$$

subject to the PC constraint (14) and the IC constraint (15),

where $i \neq j \in \{1,2\}$, $\theta_{i,t}^C \equiv \theta_i^C(z_t)$, and $\Pi_i(\theta_{i,t}, \theta_{j,t})$ is defined in (9).

The optimal default boundaries, $\underline{z}_1^C(z_{2,t})$ and $\underline{z}_2^C(z_{1,t})$, are endogenously determined by the following value matching and smooth pasting conditions:

$$E_i^C(z_t)\Big|_{z_{i,t}=\underline{z}_i^C(z_{j,t})} = 0 \quad \text{and} \quad \frac{\partial}{\partial z_{i,t}} E_i^C(z_t)\Big|_{z_{i,t}=\underline{z}_i^C(z_{j,t})} = 0, \text{ respectively.} \qquad (17)$$

The boundary condition at $z_{i,t} \to +\infty$ is identical to that in the non-collusive equilibrium. This is because when $z_{i,t} \to +\infty$, firm i is essentially an industry monopoly and there is no benefit from collusion.

[2] One interpretation of $N_{i,t}$ is that, with a probability of $1 - \xi \mathrm{d}t$ over $[t, t+\mathrm{d}t)$, the deviator can persuade its rival not to enter the non-collusive equilibrium over $[t, t+\mathrm{d}t)$. Ex-post renegotiations can occur because the non-collusive equilibrium is not renegotiation-proof or "immune to collective rethinking" (Farrell and Maskin, 1989). The strategy we consider is essentially a probabilistic punishment strategy. This "inertia assumption" also solves the technical issue of continuous-time dynamic games about the indeterminacy of outcomes (e.g., Simon and Stinchcombe, 1989; Bergin and MacLeod, 1993).

Equilibrium Deviation Values. We denote by $\underline{z}_i^D(z_{j,t})$ firm i's endogenous default boundary with respect to $z_{i,t}$ if firm i deviates from collusion and its rival j continues to follow the collusive profit-margin-setting scheme. Conditional on $z_{i,t} > \underline{z}_i^D(z_{j,t})$ for $i \neq j \in \{1,2\}$, the value functions $E_i^D(z)$ satisfy the following coupled HJB equations:

$$r_f E_i^D(z_t)\mathrm{d}t = \max_{\theta_{i,t}} (1-\tau)[\Pi_i(\theta_{i,t}, \theta_{j,t}^C)e^{z_{i,t}} - e^{b_i}]\mathrm{d}t \\ - \xi \left[E_i^D(z_t) - E_i^N(z_t)\right]\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}E_i^D(z_t)], \tag{18}$$

where $i \neq j \in \{1,2\}$, $\theta_{i,t}^C \equiv \theta_i^C(z_t)$, and $\Pi_i(\theta_{i,t}, \theta_{j,t})$ is defined in (9).

The optimal default boundaries, $\underline{z}_1^D(z_{2,t})$ and $\underline{z}_2^D(z_{1,t})$, are endogenously determined by the value matching and smooth pasting conditions:

$$E_i^D(z_t)\Big|_{z_{i,t}=\underline{z}_i^D(z_{j,t})} = 0 \quad \text{and} \quad \frac{\partial}{\partial z_{i,t}} E_i^D(z_t)\Big|_{z_{i,t}=\underline{z}_i^D(z_{j,t})} = 0, \quad \text{respectively.} \tag{19}$$

The boundary condition at $z_{i,t} \to +\infty$ is identical to that in the non-collusive equilibrium.

Two points require discussion. First, the PC constraint (14) can become binding in the collusive equilibrium, triggering the two firms to switch to the non-collusive equilibrium. The endogenous switch captures the endogenous outbreak of price wars. We assume that once the two firms switch to the non-collusive equilibrium, they will stay there forever.[3] Endogenous switching from the collusive to the non-collusive equilibrium because of increased financial distress (i.e., higher leverage ratios) is one of our model's key differences from that of Dou, Ji and Wu (2021*a*,*b*), in which firms are financed wholly by equity and never suffer from financial distress. In their model, the PC constraint is never binding because higher profit margins always lead to higher equity values in the absence of financial distress costs owing to costly default or exit.

Second, there exist infinitely many elements in the set of incentive compatible collusion agreements, $\mathcal{C}$, and hence infinitely many collusive equilibria. We focus on a subset of $\mathcal{C}$, denoted by $\overline{\mathcal{C}}$, consisting of all profit-margin-setting schemes $\Theta^C(\cdot)$ such that the IC constraints (15) are binding state by state, that is, $E_i^D(z_t) = E_i^C(z_t)$ for all $z_t \in \mathcal{Z}$ and $i \in \{1,2\}$.[4] The subset $\overline{\mathcal{C}}$ is nonempty because it contains the profit-margin-setting scheme in the non-collusive equilibrium. We further narrow our focus to the "Pareto-efficient frontier" of $\overline{\mathcal{C}}$, denoted by $\overline{\mathcal{C}}_p$, consisting of all pairs of $\Theta^C(\cdot)$ such that there does not exist

[3]As the firm that proposes switching to the non-collusive equilibrium is essentially deviating, we assume that the two firms will not return to the collusive equilibrium. We make this assumption to be consistent with our specification for the punishment strategy.

[4]This equilibrium refinement is similar in spirit to Abreu (1988), Alvarez and Jermann (2000, 2001), and Opp, Parlour and Walden (2014)

another pair $\widetilde{\Theta}^C(\cdot) = (\widetilde{\theta}_1^C(z_t), \widetilde{\theta}_2^C(z_t)) \in \overline{\mathcal{C}}$ such that such that the implied firm values are higher for all $z_t \in \mathcal{Z}$ and $i \in \{1,2\}$, with strict inequality held for some i and z_t. Deviation never occurs on the equilibrium path. The one-shot deviation principle (Fudenberg and Tirole, 1991) makes it clear that the collusive equilibrium characterized above is subgame perfect.

Debt Value. The debt value equals the sum of the present value of cash flows that accrue to debtholders until the occurrence of an endogenous default or an idiosyncratic left-tail jump shock (i.e., exogenous displacement), whichever occurs first, plus the recovery value. We follow the literature on dynamic debt models (e.g., Mello and Parsons, 1992; Leland, 1994; Hackbarth, Miao and Morellec, 2006) and set the recovery value of endogenous default to a fraction $\delta \in (0,1)$ of the firm's unlevered asset value, $A_i^C(z_t)$, which is the value of an all-equity firm. In the collusive equilibrium, the unlevered asset value $A_i^C(z_t)$ is similarly determined by equations (12) to (19) with the IC and PC constraints satisfied, except that we set $e^{b_i} = 0$ and remove the default boundary conditions (13), (17), and (19).

The value of debt in the non-default region of the collusive equilibrium (i.e., $z_{i,t} > \underline{z}_i^C(z_{j,t})$ for $i \neq j$), denoted by $D_i^C(z_t)$, can be characterized by the following coupled HJB equations:

$$r_f D_i^C(z_t)\mathrm{d}t = e^{b_i}\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}D_i^C(z_t)], \quad \text{for } i = 1,2, \tag{20}$$

with the following boundary conditions:

$$D_i^C(z_t)\Big|_{z_{i,t}=\underline{z}_i^C(z_{j,t})} = \delta A_i^C(z_t)\Big|_{z_{i,t}=\underline{z}_i^C(z_{j,t})} \text{ and } \lim_{z_{i,t}\to+\infty} D_i^C(z_t) = \frac{e^{b_i}}{r_f + \nu}. \tag{21}$$

In equation (21), the first condition is the recover value to debtholders at the default boundary. The second condition captures the asymptotic behavior of debt when $z_{i,t} \to \infty$; in this case, the default risk of firm i arises only from the idiosyncratic left-tail jump shock, which occurs at a rate of ν.

Optimal Coupon Choice. We now illustrate the optimal initial coupon choice of firms at $t = 0$. The log coupon rates b_1 and b_2 are optimally determined in the Nash equilibrium. The best response function $\mathbf{b_i}(b_j)$ is defined as follows:

$$\mathbf{b_i}(b_j) \equiv \underset{b_i}{\operatorname{argmax}}\, V_i^C(z_0; b_i, b_j), \quad \text{with } i \neq j \in \{1,2\}. \tag{22}$$

The initial firm value is $V_i^C(z_0; b_i, b_j) \equiv E_i^C(z_0; b_i, b_j) + D_i^C(z_0; b_i, b_j)$, where the notations of $E_i^C(\cdot)$ and $D_i^C(\cdot)$ are slightly changed to explicitly highlight their dependence on (b_i, b_j).

The equilibrium leverage (b_1^C, b_2^C) is determined by the following condition:

$$b_i^C = \mathbf{b_i}(b_j^C), \quad \text{for } i \neq j \in \{1,2\}. \tag{23}$$

1.3 Motivation

Our investigation of the competition-distress feedback effect is motivated by the theoretical literature that emphasizes the feedback effects between the capital market and the real economy. For example, seminal works of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) show that the price-dependent financing constraints can spur an adverse feedback loop. Bond, Edmans and Goldstein (2012) emphasize that the fundamental-based feedback effect mainly concerns primary financial markets, whereas the equally crucial feedback effect between secondary financial markets and the real economy is mainly transmitted through the information-based channel (e.g., Chen, Goldstein and Jiang, 2006; Bakke and Whited, 2010; Edmans, Goldstein and Jiang, 2012).

Our study is also motivated by the large and growing literature that studies how firms' decisions and characteristics in financial markets interact with those in product markets. Theoretical works on this topic are, e.g., Maksimovic and Titman (1991), Bolton and Scharfstein (1990), and Hackbarth, Mathews and Robinson (2014). Empirical works, on the one hand, study the implications of financial frictions for product markets, e.g., Phillips (1995), Hortaçsu et al. (2013), Koijen and Yogo (2015), Hoberg and Phillips (2016), Gilchrist et al. (2017), Hackbarth and Taub (2018), Banerjee et al. (2019), and Grieser and Liu (2019). On the other hand, there are empirical works investigating how product market competition affects various corporate policies, e.g., Kovenock and Phillips (1997), Banerjee, Dasgupta and Kim (2008), Hoberg and Phillips (2010), and Hoberg, Phillips and Prabhala (2014).

2 Boundary Conditions

2.1 Endogenous Default Boundary Conditions

The state space is $(z_1, z_2) \in \mathcal{Z} \subset \mathbb{R}^2$. We denote the default boundary of firm $i \in \{1,2\}$ in the non-collusive equilibrium by $\mathcal{T}_i^N$, which is a 1-dimensional manifold in $\mathbb{R}^2$. The idea is the same for the default boundaries in the collusive equilibrium and in the characterization of the deviation value. Put simply, the default boundary $\mathcal{T}_i^N$ is a "curve" in $\mathbb{R}^2$. In general, the value matching condition for the default boundary of firm i is

$$E_i^N(z_1,z_2)\Big|_{(z_1,z_2)\in\mathcal{T}_i^N}=0,\quad \text{for } i\in\{1,2\}, \tag{24}$$

and the smooth pasting conditions are

$$\frac{\partial}{\partial z_i}E_i^N(z_1,z_2)\Big|_{(z_1,z_2)\in\mathcal{T}_i^N}=0 \quad\text{and}\quad \frac{\partial}{\partial z_j}E_i^N(z_1,z_2)\Big|_{(z_1,z_2)\in\mathcal{T}_i^N}=0,\quad \text{for } i\neq j\in\{1,2\}. \tag{25}$$

For the default boundary $\mathcal{T}_i^N$, we can conveniently parameterize it using z_j because there is always a threshold point $\underline{z}_i^N(z_j)$ below which the shareholders of firm i would choose to default on the debt for any $z_j\in\mathbb{R}$. Thus, the default boundary $\mathcal{T}_i^N$ can naturally be represented by the function $\underline{z}_i^N(z_j)$. As a result, the value matching condition in (24) can be rewritten as

$$E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}=0,\quad \forall\, z_j,\quad \text{for } i\neq j\in\{1,2\}, \tag{26}$$

and the smooth pasting conditions in (25) can be rewritten as

$$\frac{\partial}{\partial z_i}E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}=0 \quad\text{and}\quad \frac{\partial}{\partial z_j}E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}=0,\quad \forall\, z_j,\quad \text{for } i\neq j\in\{1,2\}. \tag{27}$$

Obviously, because the specific structure of the default boundary, the second smooth pasting condition in (27) is redundant given that the first smooth pasting condition in (27) and the value matching condition in (26) hold simultaneously for arbitrary $z_j\in\mathbb{R}$. More precisely, according to the chain rule, the value matching condition in (26) implies that

$$\frac{\partial}{\partial z_i}E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}\times\frac{\partial}{\partial z_j}\underline{z}_i^N(z_j)\;+\;\frac{\partial}{\partial z_j}E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}=0,\ \forall\, z_j,\quad \text{for }\ i\neq j\in\{1,2\}, \tag{28}$$

where the first term captures the indirect effect of a small local change of z_j on $E_i^N(z_1,z_2)$ through its impact on the default boundary $\underline{z}_i^N(z_j)$, and the second term captures the direct effect of a small local change of z_j on $E_i^N(z_1,z_2)$; further, the relation established in (28) in turn leads to

$$\frac{\partial}{\partial z_j}E_i^N(z_1,z_2)\Big|_{z_i=\underline{z}_i^N(z_j)}=0,\quad \forall\, z_j,\quad \text{for } i\neq j\in\{1,2\}, \tag{29}$$

because the first smooth pasting condition in (27) implies the following equality:

$$\frac{\partial}{\partial z_i} E_i^N(z_1, z_2)\bigg|_{z_i = \underline{z}_i^N(z_j)} \times \frac{\partial}{\partial z_j} \underline{z}_i^N(z_j) = 0, \ \forall\, z_j, \ \text{ for } \ i \neq j \in \{1, 2\}. \tag{30}$$

Although the state space is 2-dimensional (i.e., there are two state variables z_1 and z_2), there is only one "free boundary" to be characterized, which can be parameterized by one state variable. In general, the free boundary of an optimal stopping problem with K state variables where $K \geq 2$ (i.e., a free boundary optimal control problem with K state variables), as a $(K-1)$-dimensional manifold in the K-dimensional state space, often cannot be parameterized by any $(K-1)$-subset of the state variables; in such cases, smooth pasting conditions in all different directions are needed to describe the free boundary conditions properly, similar in spirit to (24) and (25). For example, Chen et al. (2022) and Kakhbod et al. (2022) consider corporate liquidity models with multiple state variables in which there is an economic boundary condition for each of the state variables. Their free boundary optimal control problems, by nature, are more complex than ours here, and as a result, they cannot simplify the smooth pasting conditions by ignoring some redundant ones, similar to what we do.

2.2 Boundary Conditions at Infinity

When $z_{i,t} = +\infty$, firm i essentially monopolizes the industry with negligible financial leverage because its competitor, firm j, has a negligible size regardless of its $z_{j,t}$. Thus, the boundary condition of firm i's equity value at $z_{i,t} = +\infty$ should satisfy

$$\lim_{z_{i,t} \to \infty} \frac{\partial}{\partial z_{i,t}} E_i^N(z_t) = \lim_{z_{i,t} \to \infty} \frac{\partial}{\partial z_{i,t}} E_i^C(z_t) = \lim_{z_{i,t} \to \infty} \frac{\partial}{\partial z_{i,t}} E_i^D(z_t) = \lim_{z_{i,t} \to \infty} \frac{\partial}{\partial z_{i,t}} U_i(z_{i,t}), \tag{31}$$

where $U_i(z_{i,t})$ is the equity value of an unlevered monopoly firm with $z_{i,t} \equiv a_t$ and price $P_{i,t} \equiv P_t$. In this industry, the demand curve facing the monopoly firm is $e^{z_{i,t}} C_{i,t}$, where $C_{i,t}$ given by equation (2), i.e.,

$$C_{i,t} \equiv C_t = P_{i,t}^{-\epsilon}, \tag{32}$$

and the evolution of $z_{i,t}$ is given by equation (8),

$$e^{-z_{i,t}} \mathrm{d} e^{z_{i,t}} = (g - \varsigma\gamma)\mathrm{d}t + \varsigma \mathrm{d}W_t^{\mathbb{Q}} + \sigma \mathrm{d}W_{i,t} - \mathrm{d}J_{i,t}. \tag{33}$$

Thus, the HJB equation that determines $U(z_{i,t})$ can be written as

$$r_f U_i(z_{i,t})\mathrm{d}t = \max_{\theta_{i,t}} \quad (1-\tau)\left[\omega^{1-\epsilon}\theta_{i,t}(1-\theta_{i,t})^{\epsilon-1}e^{z_{i,t}} - e^{b_i}\right]\mathrm{d}t + \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{d}U_i(z_{i,t})\right]. \tag{34}$$

3 Numerical Algorithm

To give an overview, our algorithm proceeds in the following steps:

(1). We solve the non-collusive equilibrium. This requires us to solve the subgame perfect equilibrium of the dynamic game played by two firms. The simultaneous-move dynamic game requires us to solve the intersection of the two firms' best response functions (i.e., optimal decisions on profit margins and default), which themselves are optimal solutions to coupled partial differential equations (PDEs).

(2). We solve the collusive equilibrium using the value functions in the non-collusive equilibrium as punishment values. Because we are interested in the highest collusive profit margins with binding IC constraints, this requires us to solve a high-dimensional fixed-points problem. Thus, we use an iteration method inspired by Abreu, Pearce and Stacchetti (1986, 1990), Ericson and Pakes (1995), and Fershtman and Pakes (2000) to solve the problem.

Note that standard methods for solving PDEs with free boundaries (e.g. finite difference or finite element) can easily lead to non-convergence of value functions. To mitigate such problems and obtain accurate solutions, we solve the continuous-time game using a discrete-time dynamic programming method, as in Dou et al. (2021); Dou and Ji (2021).

Because firm 1 and firm 2 are symmetric, one firm's equity value and policy functions are obtained directly given the other firm's equity value and policy functions. We first illustrate the non-collusive equilibrium and then the collusive equilibrium.

3.1 Non-Collusive Equilibrium

We first present the recursive formulation for the firm's equity value in the non-collusive equilibrium. Next, we present the conditions that determine the non-collusive equilibrium.

Recursive Formulation for Equity Value in Non-collusive Equilibrium. Firm i's state is characterized by two state variables, including firm i's demand intensity $z_{i,t}$ and firm j's demand intensity $z_{j,t}$. Denote by $E_i^N(z_{i,t}, z_{j,t}; b_i, b_j)$ the equity value function in the non-collusive equilibrium for $i = 1, 2$, where b_i and b_j are the two firms' log coupon rates, which will be optimally determined in the end.

To characterize the equilibrium equity value functions, it is more convenient to introduce two off-equilibrium equity value functions. Let $\widehat{E}_i^N(z_{i,t}, z_{j,t}; \theta_{j,t}, d_{j,t}; b_i, b_j)$ be firm $i(=1,2)$'s equity value when its competitor j's profit margin is any (off-equilibrium) value $\theta_{j,t}$ and default status is any (off-equilibrium) value $d_{j,t} = 0, 1$.

Firm $i = 1, 2$ solves the following problem:

$$\widehat{E}_i^N(z_{i,t}, z_{j,t}; \theta_{j,t}, d_{j,t}; b_i, b_j) = \max_{\theta_{i,t}, d_{i,t}} (1-\tau) \left[\omega^{1-\epsilon}\theta_{i,t}(1-\theta_{i,t})^{\eta-1}(1-\theta_t)^{\epsilon-\eta}e^{z_{i,t}} - e^{b_i}\right]\Delta t \quad (35)$$
$$+ (1-d_{i,t})e^{-(r_f+\nu)\Delta t}\mathbb{E}_t^{\mathbb{Q}}\left[(1-d_{j,t})E_i^N(z_{i,t+\Delta t}, z_{j,t+\Delta t}; b_i, b_j) + d_{j,t}E_i^N(z_{i,t+\Delta t}, z_{new}; b_i, b_{new})\right],$$

subject to the following constraints. (1) The industry's profit margin is given by

$$1-\theta_t = \left[\sum_{j=1}^{2} e^{z_{i,t}-a_t}\,(1-\theta_{i,t})^{\eta-1}\right]^{\frac{1}{\eta-1}} \quad \text{with} \quad e^{a_t} = e^{z_{i,t}} + e^{z_{j,t}}. \quad (36)$$

(2) Firms' demand shocks evolve according to

$$e^{z_{i,t+\Delta t}} = e^{z_{i,t}} + (g - \varsigma\gamma)e^{z_{i,t}}\Delta t + \varsigma e^{z_{i,t}}\Delta W_t^{\mathbb{Q}} + \sigma e^{z_{i,t}}\Delta W_{i,t}, \quad (37)$$
$$e^{z_{j,t+\Delta t}} = e^{z_{j,t}} + (g - \varsigma\gamma)e^{z_{j,t}}\Delta t + \varsigma e^{z_{j,t}}\Delta W_t^{\mathbb{Q}} + \sigma e^{z_{j,t}}\Delta W_{j,t}. \quad (38)$$

Non-collusive Equilibrium. Denote by $\theta_i^N(z_{i,t}, z_{j,t}; b_i, b_j)$ and $d_i^N(z_{i,t}, z_{j,t}; b_i, b_j)$ the equilibrium profit margin and default functions. Denote by $\widehat{\theta}_i^N(z_{i,t}, z_{j,t}; \theta_{j,t}, d_{j,t}; b_i, b_j)$ and $\widehat{d}_i^N(z_{i,t}, z_{j,t}; \theta_{j,t}, d_{j,t}; b_i, b_j)$ the off-equilibrium profit margin and default functions.

Given firm j's profit margin $\theta_{j,t}$ and default decision $d_{j,t}$, firm i optimally sets the profit margin $\theta_{i,t}$ and makes default decision $d_{i,t}$. The non-collusive equilibrium is derived from the fixed point—each firm's profit margin and default are optimal given the other firm's optimal profit margin and default:

$$\theta_i^N(z_{i,t}, z_{j,t}; b_i, b_j) = \widehat{\theta}_i^N(z_{i,t}, z_{j,t}; \theta_j^N(z_{j,t}, z_{i,t}; b_j, b_i), d_j^N(z_{j,t}, z_{i,t}; b_j, b_i); b_i, b_j), \quad (39)$$
$$d_i^N(z_{i,t}, z_{j,t}; b_i, b_j) = \widehat{d}_i^N(z_{i,t}, z_{j,t}; \theta_j^N(z_{j,t}, z_{i,t}; b_j, b_i), d_j^N(z_{j,t}, z_{i,t}; b_j, b_i); b_i, b_j). \quad (40)$$

The equilibrium equity value functions are given by

$$E_i^N(z_{i,t}, z_{j,t}; b_i, b_j) = \widehat{E}_i^N(z_{i,t}, z_{j,t}; \theta_j^N(z_{j,t}, z_{i,t}; b_j, b_i), d_j^N(z_{j,t}, z_{i,t}; b_j, b_i); b_i, b_j). \quad (41)$$

3.2 Collusive Equilibrium

We first present the recursive formulation for the firm's equity value in the collusive equilibrium. Next, we present the recursive formulation for the firm's equity value when it deviates from the collusive equilibrium. Finally, we present the IC constraints to determine the equilibrium collusive profit margins. After finding the equilibrium collusive profit margin scheme, we check whether the PC constraints are satisfied. There are two cases, if the PC constraints are satisfied, the two firms will collude on the equilibrium profit margin scheme. If the PC constraints are not satisfied, the two firms will set profit margins according to their non-collusive ones.

Recursive Formulation for Equity Value in The Collusive Equilibrium. Denote by $\overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot))$ firm i's equity value in the collusive equilibrium with collusive profit margin scheme $\overline{\Theta}^C(\cdot)$, for $i = 1, 2$. Denote by $\widehat{E}_i^C(z_{i,t}, z_{j,t}; d_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot))$ firm i's equity value in the collusive equilibrium with collusive profit margin scheme $\overline{\Theta}^C(\cdot)$ when its competitor j's default status is any (off-equilibrium) value $d_{j,t} = 0, 1$.

Firm i solves the following problem:

$$\begin{aligned}\widehat{E}_i^C(z_{i,t}, z_{j,t}; d_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) = \max_{d_{i,t}} (1-\tau) \left[\omega^{1-\epsilon}\overline{\theta}_{i,t}^C(1-\overline{\theta}_{i,t}^C)^{\eta-1}(1-\overline{\theta}_t^C)^{\epsilon-\eta}e^{z_{i,t}} - e^{b_i}\right]\Delta t \\ + (1-d_{i,t})e^{-(r_f+v)\Delta t}\mathbb{E}_t^Q\left[(1-d_{j,t})\overline{E}_i^C(z_{i,t+\Delta t}, z_{j,t+\Delta t}; b_i, b_j; \overline{\Theta}^C(\cdot)) + d_{j,t}\overline{E}_i^C(z_{i,t+\Delta t}, z_{new}; b_i, b_{new}; \overline{\Theta}^C(\cdot))\right],\end{aligned} \tag{42}$$

subject to the following constraints. (1) The industry's profit margin is given by

$$1 - \overline{\theta}_t^C = \left[\sum_{j=1}^{2} e^{z_{i,t}-a_t}\left(1-\overline{\theta}_{i,t}^C\right)^{\eta-1}\right]^{\frac{1}{\eta-1}} \quad \text{with} \quad e^{a_t} = e^{z_{i,t}} + e^{z_{j,t}}. \tag{43}$$

(2) Firms' demand shocks evolve according to

$$e^{z_{i,t+\Delta t}} = e^{z_{i,t}} + (g - \varsigma\gamma)e^{z_{i,t}}\Delta t + \varsigma e^{z_{i,t}}\Delta W_t^Q + \sigma e^{z_{i,t}}\Delta W_{i,t}, \tag{44}$$

$$e^{z_{j,t+\Delta t}} = e^{z_{j,t}} + (g - \varsigma\gamma)e^{z_{j,t}}\Delta t + \varsigma e^{z_{j,t}}\Delta W_t^Q + \sigma e^{z_{j,t}}\Delta W_{j,t}. \tag{45}$$

Denote by $\overline{d}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot))$ the equilibrium default function. Denote by $\widehat{d}_i^C(z_{i,t}, z_{j,t}; d_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot))$ the off-equilibrium default function. The default decisions are determined in Nash equilibrium. In particular, given firm j's default decision $d_{j,t}$, firm i optimally makes default decision $d_{i,t}$. The Nash equilibrium is derived from the fixed point—each firm's default is optimal given the other firm's optimal default:

$$\overline{d}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) = \widehat{d}_i^C(z_{i,t}, z_{j,t}; \overline{d}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \overline{\Theta}^C(\cdot)); b_i, b_j; \overline{\Theta}^C(\cdot)). \tag{46}$$

The equilibrium equity value functions are given by

$$\overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) = \widehat{E}_i^C(z_{i,t}, z_{j,t}; \overline{d}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \overline{\Theta}^C(\cdot)); b_i, b_j; \overline{\Theta}^C(\cdot)). \tag{47}$$

Recursive Formulation for Equity Value upon Deviation. The deviation equity value is obtained by assuming that firm i optimally sets its profit margin conditional on firm j setting the profit margin according to the collusive profit margin scheme, i.e., $\overline{\theta}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \overline{\Theta}^C(\cdot))$ and default decision $\overline{d}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \overline{\Theta}^C(\cdot))$.

Denote by $\overline{E}_i^D(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot))$ firm i's deviation equity value. Firm i solves the following problem:

$$\begin{aligned}
\overline{E}_i^D(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) = \max_{\theta_{i,t}, d_{i,t}} & (1-\tau)\left[\omega^{1-\epsilon}\theta_{i,t}(1-\theta_{i,t})^{\eta-1}(1-\overline{\theta}_t^D)^{\epsilon-\eta}e^{z_{i,t}} - e^{b_i}\right]\Delta t \\
& + (1-d_{i,t})e^{-(r_f+\nu)\Delta t}\mathbb{E}_t^Q\Big[d_{j,t}\Big((1-\xi\Delta t)\overline{E}_i^D(z_{i,t+\Delta t}, z_{new}; b_i, b_{new}; \overline{\Theta}^C(\cdot)) \\
& + \xi\Delta t E_i^N(z_{i,t+\Delta t}, z_{new}; b_i, b_{new})\Big) + (1-d_{j,t})\Big(\xi\Delta t E_i^N(z_{i,t+\Delta t}, z_{j,t+\Delta t}; b_i, b_j) \\
& + (1-\xi\Delta t)\overline{E}_i^D(z_{i,t+\Delta t}, z_{j,t+\Delta t}; b_i, b_j; \overline{\Theta}^C(\cdot))\Big)\Big],
\end{aligned} \tag{48}$$

subject to the following constraints. (1) The industry's profit margin is given by

$$1-\overline{\theta}_t^D = \left[e^{z_{i,t}-a_t}(1-\theta_{i,t})^{\eta-1} + e^{z_{j,t}-a_t}\left(1-\overline{\theta}_{j,t}^C\right)^{\eta-1}\right]^{\frac{1}{\eta-1}} \quad \text{with} \quad e^{a_t} = e^{z_{i,t}} + e^{z_{j,t}}. \tag{49}$$

(2) Firms' demand shocks evolve according to

$$e^{z_{i,t+\Delta t}} = e^{z_{i,t}} + (g-\varsigma\gamma)e^{z_{i,t}}\Delta t + \varsigma e^{z_{i,t}}\Delta W_t^Q + \sigma e^{z_{i,t}}\Delta W_{i,t}, \tag{50}$$

$$e^{z_{j,t+\Delta t}} = e^{z_{j,t}} + (g-\varsigma\gamma)e^{z_{j,t}}\Delta t + \varsigma e^{z_{j,t}}\Delta W_t^Q + \sigma e^{z_{j,t}}\Delta W_{j,t}. \tag{51}$$

Solving for Equilibrium Profit Margins. The collusive equilibrium is a subgame perfect Nash equilibrium if and only if the collusive profit margin scheme $\overline{\Theta}^C(\cdot)$ satisfies the following PC and IC constraints:

$$\overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) \geq E_i^N(z_{i,t}, z_{j,t}; b_i, b_j), \tag{52}$$

$$\overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)) \geq \overline{E}_i^D(z_{i,t}, z_{j,t}; b_i, b_j; \overline{\Theta}^C(\cdot)), \tag{53}$$

for all $z_{i,t}$ and $i = 1, 2$.

There exist infinitely many subgame perfect Nash equilibria. We focus on the collusive equilibrium with the collusive profit margins lying on the "Pareto efficient frontier" (denoted by $\Theta^C(\cdot)$), which are obtained when all IC constraints are binding, i.e.,

$$\overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)) = \overline{E}_i^D(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{54}$$

for all $z_{i,t}$ and $i = 1, 2$. The collusive equilibrium is solved by finding the collusive profit margin scheme $\Theta^C(\cdot)$ such that the PC constraint (52) and the IC constraint (54) are satisfied simultaneously.

We denote $E_i^C(z_{i,t}, z_{j,t}; b_i, b_j)$ as firm i's value in the collusive equilibrium with the collusive profit margin scheme $\Theta^C(\cdot)$. In solving the equilibrium, we first ignore the PC constraint (52) and solve for $\Theta^C(\cdot)$ that satisfies the IC constraint (54). Then given $\Theta^C(\cdot)$, for each value of $z_{i,t}$ and $z_{j,t}$, we check whether the PC constraint (52) is satisfied for both i and j. If it is satisfied, the equity value and profit margin in the collusive equilibrium are determined according to the collusive value

$$E_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{55}$$

$$\theta_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{\theta}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{56}$$

$$E_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{E}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)), \tag{57}$$

$$\theta_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{\theta}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)). \tag{58}$$

If it is not satisfied, there are two cases. First, if firm i's PC constraint is not satisfied, then we search for the endogenous collusion boundary $\lambda_i(z_{j,t}; b_i, b_j)$ at which firm i's PC constraint just becomes binding. Then, given $z_{j,t}$, for all $z_{i,t} \leq \lambda_i(z_{j,t}; b_i, b_j)$, the equity values and profit margins in the collusive equilibrium are determined according to the non-collusive value

$$E_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = E_i^N(z_{i,t}, z_{j,t}; b_i, b_j), \tag{59}$$

$$\theta_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \theta_i^N(z_{i,t}, z_{j,t}; b_i, b_j), \tag{60}$$

$$E_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = E_j^N(z_{j,t}, z_{i,t}; b_j, b_i), \tag{61}$$

$$\theta_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \theta_j^N(z_{j,t}, z_{i,t}; b_j, b_i). \tag{62}$$

For all $z_{i,t} > \lambda_i(z_{j,t}; b_i, b_j)$, the equity values and profit margins in the collusive equilibrium are determined according to the collusive value

$$E_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{63}$$

$$\theta_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{\theta}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{64}$$

$$E_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{E}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)). \tag{65}$$

$$\theta_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{\theta}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)). \tag{66}$$

Second, if firm j's PC constraint is not satisfied, then we search for the endogenous collusion boundary $\lambda_j(z_{i,t}; b_j, b_i)$ at which firm j's PC constraint just becomes binding. Then, given $z_{i,t}$, for all $z_{j,t} \leq \lambda_j(z_{i,t}; b_j, b_i)$, the equity values and profit margins in the collusive equilibrium are determined according to the non-collusive value

$$E_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = E_i^N(z_{i,t}, z_{j,t}; b_i, b_j), \tag{67}$$

$$\theta_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \theta_i^N(z_{i,t}, z_{j,t}; b_i, b_j), \tag{68}$$

$$E_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = E_j^N(z_{j,t}, z_{i,t}; b_j, b_i), \tag{69}$$

$$\theta_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \theta_j^N(z_{j,t}, z_{i,t}; b_j, b_i). \tag{70}$$

For all $z_{j,t} > \lambda_j(z_{i,t}; b_j, b_i)$, the equity values and profit margins in the collusive equilibrium are determined according to the collusive value

$$E_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{E}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{71}$$

$$\theta_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = \overline{\theta}_i^C(z_{i,t}, z_{j,t}; b_i, b_j; \Theta^C(\cdot)), \tag{72}$$

$$E_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{E}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)). \tag{73}$$

$$\theta_j^C(z_{j,t}, z_{i,t}; b_j, b_i) = \overline{\theta}_j^C(z_{j,t}, z_{i,t}; b_j, b_i; \Theta^C(\cdot)). \tag{74}$$

Value of Debt. Firm i's debt value in the collusive equilibrium is given by

$$\begin{aligned} D_i^C(z_{i,t}, z_{j,t}; b_i, b_j) = & e^{b_i}\Delta t + (1 - d_{i,t}^C) e^{-(r_f+\nu)\Delta t} \mathbb{E}_t^Q \Big[(1 - d_{j,t}^C) D_i^C(z_{i,t+\Delta t}, z_{j,t+\Delta t}; b_i, b_j) \\ & + d_{j,t}^C D_i^C(z_{i,t+\Delta t}, z_{new}; b_i, b_{new}) \Big] \Big\} + d_{i,t}^C \nu A_i^C(z_{i,t}, z_{j,t}; b_j), \end{aligned} \tag{75}$$

where $d_{i,t}^C \equiv d_i^C(z_{i,t}, z_{j,t}; b_i, b_j)$ and $A_i^C(z_{i,t}, z_{j,t}; b_j)$ for $i = 1, 2$ are the optimal default decision and the unlevered asset value in the collusive equilibrium under the collusive profit margin scheme $\Theta^C(\cdot)$. Firm i's debt value in the non-collusive equilibrium is determined similarly using the optimal default decision and the unlevered asset value in the non-collusive equilibrium.

Optimal Choice of Debt. When firm i enters the market at time t_0, it optimally chooses b_i to maximize equity value (which is equal to the firm value after debt issuance) by solving the following problem:

$$\max_{b_i} = E_i^C(z_{i,t_0}, z_{j,t_0}; b_i, b_j) + D_i^C(z_{i,t_0}, z_{j,t_0}; b_i, b_j), \tag{76}$$

where the competitor's log coupon rate b_j is given because firms can choose coupon rates only at the beginning, when they enter the market.

At $t = 0$, we need to solve a fixed point problem in terms of b_i and b_j because the two firms choose coupon rates simultaneously. Moreover, because the two firms are symmetric at the very beginning ($z_{i,0} = z_{j,0}$), the initial optimal debt choice is also the same, i.e., $b_i = b_j$.

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