# A Note on Additional Materials for "Misallocation and Asset Prices"

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**Abstract**

This note includes additional materials for the paper titled "Misallocation and Asset Prices" (Dou et al., 2024). Section 1 provides supplemental material for the model. Section 2 provides supplemental material for empirical analyses. Section 3 describes the numerical algorithm that solves the model.

## 1 Supplemental Material for Model

### 1.1 Discussions for the Parametric Approximation of the Distribution of $\ln a_{i,t}$

We provide a heuristic discussion that the actual distribution of $\widetilde{a}_{i,t} \equiv \ln a_{i,t}$ is approximately a normal distribution. In the absence of aggregate shocks, consider the deterministic balanced growth path. Thus, all equilibrium prices are constant as shown in Online Appendix 3.7. The productivity cutoff $\underline{z}$ determined by equation (IA.19) in the online appendix becomes:

$$\underline{z}\kappa = r_f + \delta_k. \tag{ON.1}$$

Rewrite equations (2) and (21) in the main text using (ON.1) as follows:

$$\frac{\mathrm{d}a_{i,t}}{\mathrm{d}t} = s(z_{i,t})a_{i,t}, \tag{ON.2}$$

where

$$s(z) = (1+\lambda)\kappa \max\{z - \underline{z}, 0\} + r_f - \rho - \delta_a, \tag{ON.3}$$

and $\kappa$ is given by equation (IA.79) in the online appendix.

To better illustrate intuitions, we rewrite (ON.2) in discrete time with a time interval $\Delta t \approx 0$:

$$a_{i,t+\Delta t} = [1 + s(z_{i,t})\Delta t]\, a_{i,t}. \tag{ON.4}$$

We denote $a_{i,n} \equiv a_{i,n\Delta t}$ and $z_{i,n} \equiv z_{i,n\Delta t}$ for $n = 1, 2, ....$ Then, it follows that

$$a_{i,n+1} = [1 + s(z_{i,n})\Delta t]\, a_{i,n}. \tag{ON.5}$$

Define $\xi_{i,n} \equiv \ln(1 + s(z_{i,n})\Delta t) - \bar{\xi}$ with $\bar{\xi} \equiv \mathbb{E}\left[\ln(1 + s(z_{i,n})\Delta t)\right]$, and thus (ON.5) can be written as

$$\ln a_{i,n+1} = \ln a_{i,n} + \bar{\xi} + \xi_{i,n}. \tag{ON.6}$$

For a large $T > 0$, suppose we set $N_T = T/\Delta t$ (without loss of generality, we assume that $N_T$ is an integer), then equation (ON.6) implies

$$\ln a_{i,T} = \ln a_{i,1} + (N_T - 1)\bar{\xi} + \sum_{n=1}^{N_T-1} \xi_{i,n}. \tag{ON.7}$$

In the deterministic balanced growth path, $z_{i,n}$ follows a stationary process evolving according to equation (3) in the main text. Thus, the process $\xi_{i,n}$ is also stationary.

The evolution of $\ln z_{i,n}$ can be directly obtained from equation (3) in the main text, as follows:

$$\ln z_{i,n+1} = e^{-\theta\Delta t} \ln z_{i,n} + \sigma\sqrt{\frac{1 - e^{-2\theta\Delta t}}{2}}\varepsilon_{i,n+1}, \tag{ON.8}$$

where $\varepsilon_{i,n+1}$ is a standard normal variable.

According to Andrews (1983), the process $z_{i,n}$ is strong mixing with mixing coefficients dominated by an exponentially declining sequence. Let

$$\sigma_{N_T}^2 = \mathbb{E}\left[\xi_{i,1}^2\right] + 2\sum_{n=1}^{N_T-1}\left(1 - \frac{n}{N_T}\right)\mathbb{E}\left[\xi_{i,1}\xi_{i,n}\right]. \tag{ON.9}$$

Using the Berry-Esseen bound developed by Tikhomirov (1980) and Bentkus, Gotze and Tikhomoirov (1997), we obtain

$$\sup_x \left| \mathbb{P}\left\{\sum_{n=1}^{N_T-1} \xi_{i,n} \le \sigma_{N_T} x\right\} - \Phi(x)\right| \le A N_T^{-1/2} \ln^2 N_T, \tag{ON.10}$$

where $\Phi(x)$ is the CDF of a standard normal random variable, and $A$ is a constant that depends on model parameters.

### 1.2 Output Gains from Capital Reallocation

In both the data and model, capital reallocation is procyclical as the aggregate reallocation rate is higher in booms than recessions. Eisfeldt and Shi (2018) propose a method to quantify the cost of misallocation fluctuations over business cycles. The key idea of this method is to measure the potential output gain if the amount of capital reallocation observed in booms could be achieved in recessions. This method's main advantage is that it incorporates flow data on capital reallocation to help measure the cost of increased misallocation during recessions. The quantity data on flows are presumably more precisely measured than MRPK, which depends on particular model specifications.

In the model, productive firms ($z_{i,t} \geq \underline{z}_t$) lease capital from unproductive firms ($z_{i,t} < \underline{z}_t$) as shown in Lemma 1. Thus, the aggregate reallocation rate over $[t, t + \mathrm{d}t)$ is given by

$$Realloc_t = \frac{1}{A_t} \int_{\underline{z}_t}^{\infty} \int_{0}^{\infty} \lambda a \varphi_t(a, z) \mathrm{d}a\mathrm{d}z = \lambda(1 - \Omega_t(\underline{z}_t)), \tag{ON.11}$$

where $1 - \Omega_t(\underline{z}_t)$ captures the share of aggregate capital in the final goods sector held by productive firms ($z_{i,t} \geq \underline{z}_t$). Under our calibration, $\underline{z}_t$ is countercyclical, with $\mathrm{corr}(\ln \underline{z}_t, \Delta\widetilde{C}_t) \approx -0.35$. Thus, the aggregate reallocation rate $Realloc_t$ is procyclical even with a constant $\lambda$.

Using the method proposed by Eisfeldt and Shi (2018), we quantify the potential output gain in recessions if the reallocation rate of capital among firms during recessions is assumed to be as high as that during booms.[1] We apply this method to both the actual data in our 1965-2016 sample and the simulated data of our model. The estimated potential gains in recessions from capital reallocation are 3.58% and 3.09%, respectively, indicating that misallocation fluctuations have large effects on output fluctuations. The similarity in the two estimates provides a further validation of the model.

### 1.3 Budget Constraint

Consider an agent $h$ with wealth $W_t^h$ at $t$. The agent's budget constraint is

$$W_{t+\mathrm{d}t}^h = W_t^h - C_t^h \mathrm{d}t + w_t L_t^h \mathrm{d}t + (Q_{t+\mathrm{d}t} - Q_t)\mathbb{Z}_t^h + D_t \mathbb{Z}_t^h \mathrm{d}t + r_{f,t} B_t^h \mathrm{d}t, \tag{ON.12}$$

where $C_t^h \mathrm{d}t$ is the agent's consumption over $[t, t + \mathrm{d}t)$, which is assumed to be locally deterministic. The variable $w_t L_t^h \mathrm{d}t$ is the labor income over $[t, t + \mathrm{d}t)$. The variable $(Q_{t+\mathrm{d}t} - Q_t)\mathbb{Z}_t^h$ is the change in the agent's stock value, where $Q_t$ is the stock market value per share and $\mathbb{Z}_t^h$ is the number of shares held by the agent at $t$. The variable $D_t \mathbb{Z}_t^h \mathrm{d}t$ is the dividend and $r_{f,t} B_t^h \mathrm{d}t$ is the interest earnings over $[t, t + \mathrm{d}t)$, with the aggregate dividend intensity $D_t$ given by

$$D_t = \rho A_t + \int_{j=0}^{N_t} \pi_{j,t} \mathrm{d}j - S_t. \tag{ON.13}$$

The wealth $W_t^h$ consists of bonds $B_t^h$ and a share $\mathbb{Z}_t^h$ of the stock market:

$$W_t^h = Q_t \mathbb{Z}_t^h + B_t^h. \tag{ON.14}$$

Substituting equation (ON.14) into (ON.12), we obtain

$$Q_{t+\mathrm{d}t} \mathbb{Z}_{t+\mathrm{d}t}^h + B_{t+\mathrm{d}t}^h = -C_t^h \mathrm{d}t + w_t L_t^h \mathrm{d}t + Q_{t+\mathrm{d}t} \mathbb{Z}_t^h + D_t \mathbb{Z}_t^h \mathrm{d}t + (1 + r_{f,t} \mathrm{d}t) B_t^h. \tag{ON.15}$$

Aggregating equation (ON.15) over all agents, we obtain

$$C_t \mathrm{d}t + Q_{t+\mathrm{d}t} \mathbb{Z}_{t+\mathrm{d}t} + B_{t+\mathrm{d}t} = w_t L_t \mathrm{d}t + Q_{t+\mathrm{d}t} \mathbb{Z}_t + D_t \mathbb{Z}_t \mathrm{d}t + (1 + r_{f,t} \mathrm{d}t) B_t. \tag{ON.16}$$

[1] The implementation details are in Section 2.6 of this online note.

In equilibrium, the total share is normalized to be one:

$$\mathbb{Z}_t = \sum_h \mathbb{Z}_t^h \equiv 1 \;\text{ for all }\; t. \tag{ON.17}$$

Thus, equation (ON.16) becomes

$$\mathrm{d}B_t = w_t L_t \mathrm{d}t + D_t \mathrm{d}t + r_{f,t} B_t \mathrm{d}t - C_t \mathrm{d}t, \tag{ON.18}$$

which is the budget constraint. Equation (ON.18) shows that $\mathrm{d}B_t/\mathrm{d}t$ is locally deterministic because of our assumption that $C_t$ is locally deterministic.

To compute $\mathrm{d}B_t$, we use equation (IA.25) in the online appendix,

$$B_t = K_t - A_t = A_t\left(\frac{K_t}{A_t} - 1\right) = [\lambda - (1+\lambda)\Omega_t(\underline{z}_t)]A_t. \tag{ON.19}$$

Thus,

$$\mathrm{d}B_t = [\lambda - (1+\lambda)\Omega_t(\underline{z}_{t+\mathrm{d}t})]A_{t+\mathrm{d}t} - [\lambda - (1+\lambda)\Omega_t(\underline{z}_t)]A_t. \tag{ON.20}$$

The diffusion term on the right-hand side of equation (ON.20) will cancel out because $\mathrm{d}B_t$ is locally deterministic.

### 1.4 Resource Constraint

By definition, the aggregate output $Y_t \mathrm{d}t$ is

$$Y_t \mathrm{d}t = \int_{\underline{z}_t}^{\infty}\int_0^{\infty} y_t(a,z)\mathrm{d}t\varphi_t(a,z)\mathrm{d}a\mathrm{d}z = \int_0^{\infty}\int_0^{\infty} y_t(a,z)\mathrm{d}t\varphi_t(a,z)\mathrm{d}a\mathrm{d}z. \tag{ON.21}$$

Substituting equations (2) and (16) in the main text into the above equation and using (13), (17), (26), (23), (24), and (25) in the main text, we obtain

$$\begin{aligned} Y_t \mathrm{d}t =& \mathrm{d}A_t + (\delta_a \mathrm{d}t - \sigma_{a,t}\mathrm{d}W_t)A_t + w_t L_t \mathrm{d}t + (\delta_k \mathrm{d}t + \sigma_k \mathrm{d}W_t)K_t + r_{f,t}B_t \mathrm{d}t + \rho A_t \mathrm{d}t \\ &+ \int_0^{\infty}\int_0^{\infty}\left(\int_0^{N_t} p_{j,t}x_{j,t}(a,z)\mathrm{d}j\mathrm{d}t\right)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z, \end{aligned} \tag{ON.22}$$

where the last term is the revenue of the intermediate goods sector. Using equations (5) and (19) in the main text and the definition $X_t \equiv \int_{i\in\mathcal{I}} x_{i,t}\mathrm{d}i = \int_0^{\infty}\int_0^{\infty} x_t(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z$, it can be simplified as follows

$$\begin{aligned} \int_0^{\infty}\int_0^{\infty}\left(\int_0^{N_t} p_{j,t}x_{j,t}(a,z)\mathrm{d}j\mathrm{d}t\right)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z &= \int_0^{N_t}\left(\int_0^{\infty}\int_0^{\infty} p_{j,t}x_{j,t}(a,z)\varphi_t(a,z)\mathrm{d}a\mathrm{d}z\right)\mathrm{d}j\mathrm{d}t \\ &= \int_0^{N_t} p_{j,t}e_{j,t}\mathrm{d}j\mathrm{d}t \\ &= \int_0^{N_t}\pi_{j,t}\mathrm{d}j\mathrm{d}t + \int_0^{N_t} e_{j,t}\mathrm{d}j\mathrm{d}t. \end{aligned} \tag{ON.23}$$

Substituting equation (ON.23) into (ON.22), we obtain

$$Y_t \mathrm{d}t = \mathrm{d}A_t + (\delta_a \mathrm{d}t - \sigma_{a,t} \mathrm{d}W_t) A_t + w_t L_t \mathrm{d}t + (\delta_k \mathrm{d}t + \sigma_k \mathrm{d}W_t) K_t + r_{f,t} B_t \mathrm{d}t + \rho A_t \mathrm{d}t \\ + \int_0^{N_t} \pi_{j,t} \mathrm{d}j \mathrm{d}t + \int_0^{N_t} e_{j,t} \mathrm{d}j \mathrm{d}t. \tag{ON.24}$$

Substituting equation (ON.13) and the budget constraint (ON.18) into (ON.24) and using $\sigma_{a,t} = K_t / A_t \sigma_k$, we obtain the resource constraint

$$Y_t \mathrm{d}t = \underbrace{\mathrm{d}A_t + (\delta_a A_t + \delta_k K_t) \mathrm{d}t}_{\text{investment in the final goods sector}} + \underbrace{S_t \mathrm{d}t + \int_0^{N_t} e_{j,t} \mathrm{d}j \mathrm{d}t}_{\text{R\&D and intangible goods production}} \\ + C_t \mathrm{d}t + \mathrm{d}B_t. \tag{ON.25}$$

Note that the resource constraint (ON.25) holds by Walras's law in equilibrium. This can be proved by substituting equations (30) and (36) in the main text into (ON.24), and using the condition below

$$\int_0^\infty \int_0^\infty \left( \int_0^{N_t} p_{j,t} x_{j,t}(a,z) \mathrm{d}j \mathrm{d}t \right) \varphi_t(a,z) \mathrm{d}a \mathrm{d}z = \varepsilon Y_t \mathrm{d}t, \tag{ON.26}$$

which simply says that the cost of purchasing intangible goods is equal to a share $\varepsilon$ of $Y_t$ (the derivation is similar to equation (31) in the main text).

### 1.5 Inspection of Key Parameters and Mechanisms

We conduct counterfactual and sensitivity analyses to illustrate the key mechanisms of the model. Table ON.I shows how the main variables of our model respond to changes in key parameters and variables. Column (1) presents the baseline case of our full model. In column (2), we consider a less persistent idiosyncratic productivity by increasing $\theta$ from 0.1625 to 0.6931, which corresponds to a reduction in the yearly autocorrelation of $\ln z_{i,t}$ from 0.85 to 0.5. Compared with the baseline, the average misallocation $M_t$ increases from $-0.40$ to $-0.12$ because productive firms are more likely to become unproductive in the future when productivity is more transitory, weakening the self-financing channel through capital accumulation. As a result, the final goods sector's productivity $H_t$ decreases from 1.94 to 1.57. The average consumption growth rate decreases to 0.21%. A lower persistence of idiosyncratic productivity reduces the volatility of consumption growth to 1.14% and the yearly autocorrelation of consumption growth to 0.42; moreover, aggregate TFP, output, and misallocation all become less persistent. The Sharpe ratio declines from 0.39 in the baseline to 0.07 in column (2).

In column (3), we consider a more restrictive collateral constraint by reducing $\lambda$ from 1.1 to 1. The average misallocation $M_t$ remains roughly unchanged compared to the baseline. This is because the equilibrium misallocation is mainly determined by firms' differential speed of capital accumulation across different productivity $z_{i,t}$ (i.e., the term $\mathrm{Cov}(\widetilde{z}_{i,t}, \mathrm{d}\widetilde{a}_{i,t})$ in equation (38) in the main text). A change in $\lambda$ does not affect this difference much because a lower $\lambda$ scales down the revenue of both high-productivity and low-productivity firms. However, reducing $\lambda$ directly leads to a lower TFP $Z_t$ in equation (33) in the main text, reflecting the instantaneous reallocation of capital through the

Table ON.I: Inspection of key parameters.

| | (1) Baseline | (2) $\theta = 0.69$ | (3) $\lambda = 1$ | (4) $\chi = 1.3$ | (5) $\sigma_k = 0.15$ | (6) $M_t \equiv \mathbb{E}[M_t]$ |
|---|---|---|---|---|---|---|
| $\mathbb{E}[M_t]$ | −0.39 | −0.12 | −0.39 | −0.38 | −0.39 | −0.39 |
| $\mathbb{E}[H_t]$ | 1.94 | 1.57 | 1.92 | 1.95 | 1.81 | 1.63 |
| $\mathbb{E}[\Delta\widetilde{C}_t]$ (%) | 1.75 | 0.21 | 1.55 | 1.19 | 1.66 | 1.63 |
| $\sigma(\Delta\widetilde{C}_t)$ (%) | 1.66 | 1.14 | 1.52 | 1.69 | 1.27 | 0.00 |
| $AC1(\Delta\widetilde{C}_t)$ | 0.46 | 0.42 | 0.51 | 0.45 | 0.47 | 1.00 |
| $AC1(M_t)$ | 0.73 | 0.25 | 0.74 | 0.74 | 0.76 | 1.00 |
| $\sigma(M_t)$ (%) | 8.96 | 14.33 | 9.65 | 9.01 | 10.84 | 0.00 |
| Sharpe ratio | 0.39 | 0.07 | 0.39 | 0.38 | 0.27 | – |

Note: The notation $\Delta\widetilde{X}_t = \ln X_{t+1} - \ln X_t$ represents difference in $\ln X_t$ between year $t$ and year $t-1$. $AC1(\Delta\widetilde{C}_t)$ and $AC1(M_t)$ refer to the yearly autocorrelation of consumption growth and misallocation. When constructing the model moments, we simulate a sample for 1,000 years with a 100-year burn-in period, which is long enough to guarantee the stability of these moments.

capital leasing market. The lower $H_t$ reduces the average consumption growth rate to 1.55% and the volatility of consumption growth to 1.52%. The persistence of consumption growth increases from 0.46 to 0.51. The Sharpe ratio remains unchanged at 0.39 due to the offsetting effects of a lower $\sigma(\Delta\widetilde{C}_t)$ and a higher $AC1(\Delta\widetilde{C}_t)$.

In column (4), we consider a lower productivity of R&D by reducing $\chi$ from 1.35 to 1.3. Compared with our baseline in column (1), column (4) shows that all variables remain roughly unchanged, except for a lower consumption growth rate (1.19% vs. 1.75% in the baseline). The lower growth rate is determined by the productivity of R&D, rather than a better allocation of capital among firms because $H_t$ is roughly unchanged. The parameter $\chi$ plays a role of a scaling factor that determines the equilibrium growth rate.

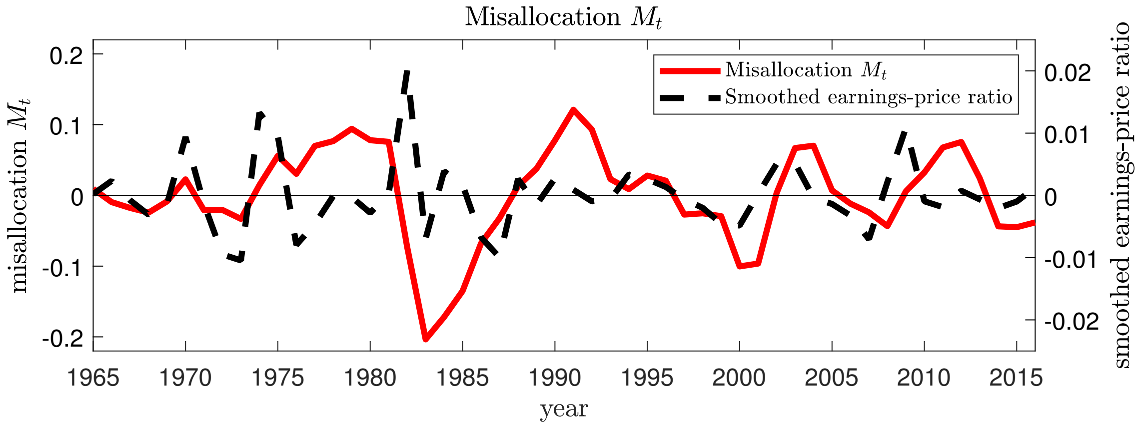
In column (5), we reduce the volatility of aggregate shocks from $\sigma_k = 0.19$ to $\sigma_k = 0.15$. This change has a negligible effect on the average level of misallocation. However, the average productivity of the final goods sector declines because aggregate risks change firms' leverage decisions and hence the aggregate $K_t/A_t$ ratio. The volatility of consumption growth declines significantly from 1.66% to 1.27% whereas the yearly autocorrelation in consumption growth remains roughly unchanged. The Sharpe ratio drops from 0.39 to 0.27 due to the lower volatility of consumption growth.

Finally, in column (6), we exogenously fix $M_t$ at its long-run mean −0.39. In this case, the volatility of consumption growth drops to zero and Sharpe ratio is not defined. It is not a surprising result because in our model, aggregate shocks affect the economy through the state variable $M_t$.

## 2 Supplemental Material for Empirical Analyses

### 2.1 Time-series of the empirical measure of misallocation $M_t$.

Figure ON.1 presents a comparison between the empirical measure of misallocation $M_t$ (illustrated by the red solid line) and the smoothed earnings-price ratio (shown as the black dashed line) introduced by Campbell and Shiller (1988). The smoothed earnings-price ratio and its variants are frequently




Note: This figure plots the time series of $M_t$ (left $y$-axis) and the smoothed earnings-price ratio (right $y$-axis) proposed by Campbell and Shiller (1988).

Figure ON.1: Time-series plot of the empirical measure of misallocation $M_t$.

employed as empirical proxies for the aggregate discount rate (e.g., Gourio, 2012; Hall, 2017; Dou, Ji and Wu, 2021, 2022). The time-series variation of this ratio typically aligns with the frequency of business cycles. Clearly, the empirical measure of misallocation, $M_t$, exhibits greater persistence compared to the smoothed earnings-price ratio, despite the two time series exhibiting positive comovement. The yearly autocorrelation of $M_t$ is 0.75, which is close to the calibrated persistence of 0.77 that Bansal and Yaron (2004) find for the predictable component of consumption growth. If misallocation $M_t$ affects economic growth, as suggested by our model, the highly persistent and volatile $M_t$ appears to capture the low-frequency growth fluctuations, referred to as the medium-term business cycle by Comin and Gertler (2006) or the growth cycle by Kung and Schmid (2015). The observed positive comovement aligns with the model-implied interaction between misallocation and the discount rate.

## 2.2 Cross-Sectional Evidence

Our theory's main implication is that the volatile and persistent time-series variation in misallocation captures the low-frequency component of the time-series variation in aggregate growth. Although our model does not analyze cross-sectional implications, we provide cross-sectional evidence to further support the theoretical mechanism. In Subsection 2.2.1, we estimate the market price of risk for the misallocation factor and study its cross-sectional asset pricing implications. In Subsection 2.2.2, we show that firms with higher book-to-market ratios are more negatively exposed to the misallocation factor, which provides further support that the slow-moving misallocation captures low-frequency growth fluctuations.

### 2.2.1 Misallocation as A Macroeconomic Risk Factor

Our model implies that the misallocation $M_t$ plays a significant role in determining the SDF of representative agent through its effects on aggregate consumption growth. To examine the empirical relevance of this mechanism, we test whether the empirical misallocation measure $M_t$ is a risk factor significantly priced in the cross section of assets.

We consider standard test assets, including 25 size-sorted and book-to-market-sorted portfolios, 10 momentum-sorted portfolios, and 6 maturity-sorted Treasury bond portfolios. For each asset $i$, we estimate the factor loadings using the following time-series regression:

$$R^e_{i,t} = c_i + \sum_k \beta_{i,k} f_{k,t} + \varepsilon_{i,t}, \tag{ON.27}$$

where $R^e_{i,t} = R_{i,t} - r_{f,t}$ is the excess return of asset $i$ over the risk-free rate and $f_{k,t}$ represents risk factor $k$. We then estimate the cross-sectional price of risk associated with the factors $f_{k,t}$ by running a cross-sectional regression of time-series average excess returns, $\mathbb{E}[R^e_{i,t}]$, on risk factor exposures estimated in equation (ON.27) as follows,

$$\mathbb{E}[R^e_{i,t}] = \alpha + \sum_k \widehat{\beta}_{i,k} \lambda_k + \epsilon_i, \tag{ON.28}$$

where the estimated $\widehat{\lambda}_k$ is the price of risk for factor $k$ and $\widehat{\alpha}$ is the average cross-sectional pricing error or zero-beta rate.

The above estimation procedure is implemented using different linear factor models. The results are presented in Table ON.II and visualized in Figures ON.2 and ON.3. As a benchmark, column (1) of Table ON.II reports the results of CAPM, which includes market excess returns as the single risk factor. It clearly shows that the exposure to market risk cannot explain the spread in average returns across portfolios. The cross-sectional intercept is statistically significant and the factor price of risk is statistically insignificant. The pricing errors are large, with a high total mean absolute pricing error (MAPE) of 2.764% and a low adjusted $R$-squared of 0.303. Column (2) of Table ON.II presents the results based on a two-factor model that includes the year-on-year changes in the empirical misallocation measure, $\Delta M_t$, as an additional risk factor. The price of risk for $\Delta M_t$ is $-0.079$, which is negative and statistically significant as implied by our model.[2] Relative to CAPM, the adjusted $R$-squared increases significantly to 0.446 and the total MAPE declines significantly to 1.465%. The test assets are lined up very close to the 45-degree line in the two factor model (Panel B of Figure ON.2), which is in sharp contrast to the prediction of CAPM (Panel A of Figure ON.2).

As another benchmark, column (3) of Table ON.II presents the results of the Fama-French three-factor (FF3) model. Comparing columns (2) and (3) of Table ON.II, the FF3 model achieves a higher adjusted $R$-squared of 0.619. However, the two-factor model with market returns and the misallocation factor $\Delta M_t$ has a lower total MAPE. The two-factor model outperforms the FF3 model especially for the 10 momentum-sorted portfolios (3.721% compared to 1.964%). It is well known that the FF3 model has a poor explanatory power for momentum-sorted portfolio returns. The cross-sectional fit is clearly displayed in Panels B and C of Figure ON.2, which shows that the two-factor model outperforms the FF3 model mainly due to the improved fit for momentum-sorted portfolios. In column (4) of Table ON.II, we further include the misallocation factor $\Delta M_t$ to the FF3 model to construct a four-factor model. Compared with the FF3 model, the cross-sectional fit further improves as shown by the lower total MAPE and higher adjusted $R$-squared in the four-factor model. The improvement is mainly due to improved explanatory power for momentum-sorted portfolio.

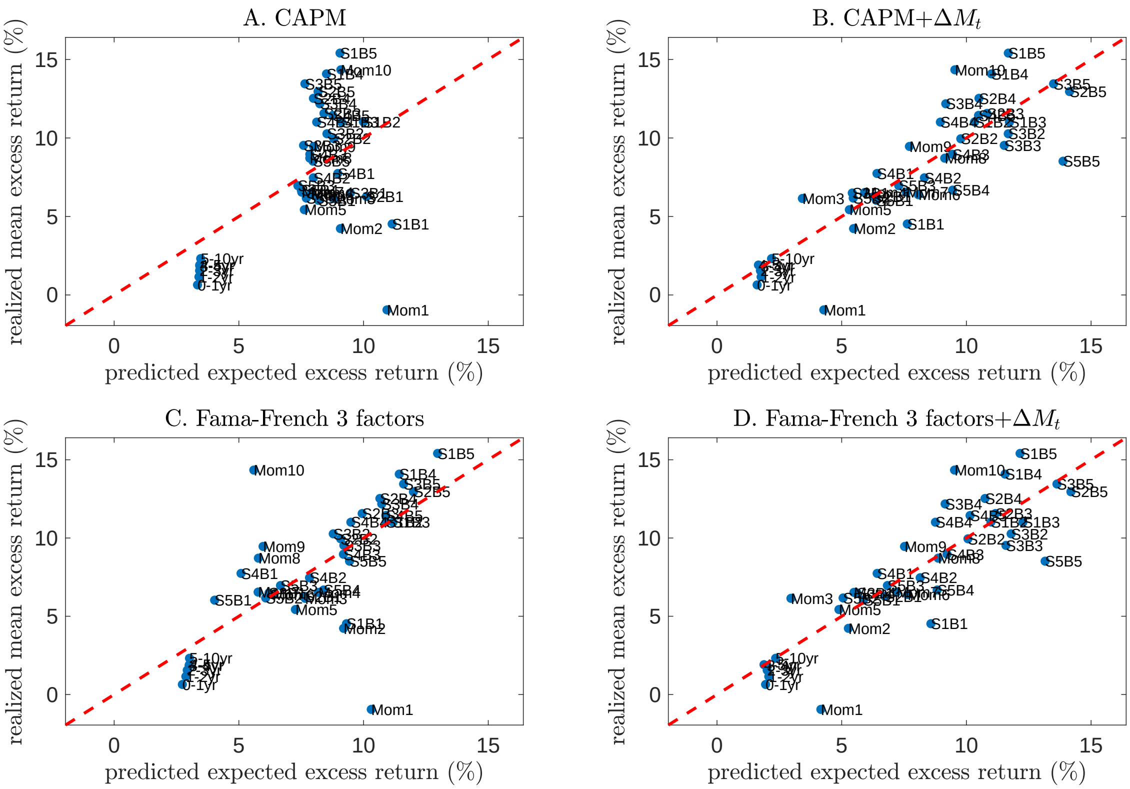
[2] The magnitude of the price of risk for $\Delta M_t$ does not represent the risk premium of $\Delta M_t$ because the misallocation factor $\Delta M_t$ does not lie in the space of excess returns.

Table ON.II: Portfolio returns and model fit.

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
|---|---|---|---|---|---|---|---|---|
| | $Mkt$ | $Mkt, \Delta M$ | $FF$ | $FF, \Delta M$ | $Mkt, CG$ | $Mkt, \Delta M, CG$ | $FF, CG$ | $FF, \Delta M, CG$ |
| | | | | Panel A: Prices of risk | | | | |
| Intercept | 3.336 | 1.425 | 2.665 | 1.781 | 2.438 | 1.412 | 2.336 | 1.464 |
| | (0.962) | (1.702) | (0.730) | (1.177) | (1.385) | (1.594) | (0.882) | (1.263) |
| $Mkt$ | 4.919 | 6.076 | 3.658 | 5.353 | 4.610 | 5.828 | 4.533 | 6.196 |
| | (3.851) | (6.237) | (3.742) | (5.589) | (4.874) | (5.750) | (4.211) | (5.903) |
| $\Delta M$ | | −0.079 | | −0.078 | | −0.070 | | −0.079 |
| | | (0.031) | | (0.030) | | (0.025) | | (0.032) |
| $SMB$ | | | 3.006 | 2.182 | | | 2.641 | 1.831 |
| | | | (2.874) | (4.307) | | | (3.232) | (4.555) |
| $HML$ | | | 4.355 | 4.804 | | | 4.396 | 4.842 |
| | | | (2.975) | (4.458) | | | (3.354) | (4.724) |
| $CG$ | | | | | 0.023 | 0.011 | 0.019 | 0.019 |
| | | | | | (0.011) | (0.013) | (0.007) | (0.011) |
| | | | | Panel B: Test diagnostics | | | | |
| Total MAPE | 2.764 | 1.465 | 1.902 | 1.450 | 2.029 | 1.393 | 1.955 | 1.469 |
| Size and B/M 25 | 2.768 | 1.527 | 1.304 | 1.472 | 1.493 | 1.401 | 1.377 | 1.576 |
| Momentum 10 | 3.304 | 1.964 | 3.721 | 1.975 | 3.617 | 1.970 | 3.713 | 1.753 |
| Bond 6 | 1.847 | 0.375 | 1.361 | 0.483 | 1.613 | 0.398 | 1.433 | 0.550 |
| Adjusted $R$-squared | 0.303 | 0.446 | 0.619 | 0.705 | 0.453 | 0.531 | 0.629 | 0.718 |

Note: This table presents pricing results for 41 test assets, including 25 size-sorted and book-to-market-sorted portfolios, 10 momentum-sorted portfolios, and 6 maturity-sorted Treasury bond portfolios. Each model is estimated using equation (ON.28). $Mkt$ is the market's excess return over the risk-free rate. $\Delta M$ is the misallocation factor, which is the year-on-year changes in the empirical misallocation measure $M_t$. $SMB$ and $HML$ are the two factors in the FF3 model, capturing the excess returns of small caps over big caps and of value stocks over growth stocks, respectively. Panel A reports the prices of risk. Shanken standard errors are reported in brackets. Panel B reports test diagnostics, including MAPE and the adjusted $R$-squared. The sample is yearly and spans the period from 1965 to 2016.

Our model suggests that the low-frequency component of aggregate consumption growth is generated by the slow-moving misallocation. If this mechanism is empirically relevant, we expect the long-run expected consumption growth to have little explanatory power for portfolio returns after including the misallocation factor $\Delta M_t$ in linear factor models. Following Parker and Julliard (2005), we use accumulated future consumption growth to approximate long-run expected consumption growth. Column (5) of Table ON.II and Panel A of Figure ON.3 show that the two-factor model with market returns and accumulated future consumption growth can fit the returns of our test portfolios well, with an adjusted $R$-squared of 0.453. In column (6) of Table ON.II and Panel B of Figure ON.3, we augment this two-factor model with the misallocation factor $\Delta M_t$ to construct a three-factor model. We find that the relation between realized mean excess returns and predicted mean excess returns across our test portfolios stays almost unchanged, implying that expected consumption growth and misallocation are indeed similarly priced in the cross section of test assets. However, the coefficient on accumulated future consumption growth becomes statistically insignificant after including $\Delta M_t$ as a factor whereas the coefficient on $\Delta M_t$ is statistically significant. Similar patterns are shown




Note: This figure plots the realized mean excess returns of portfolios against the expected excess returns predicted by various linear factor asset pricing models. The sample is yearly and spans the period from 1965 to 2016.

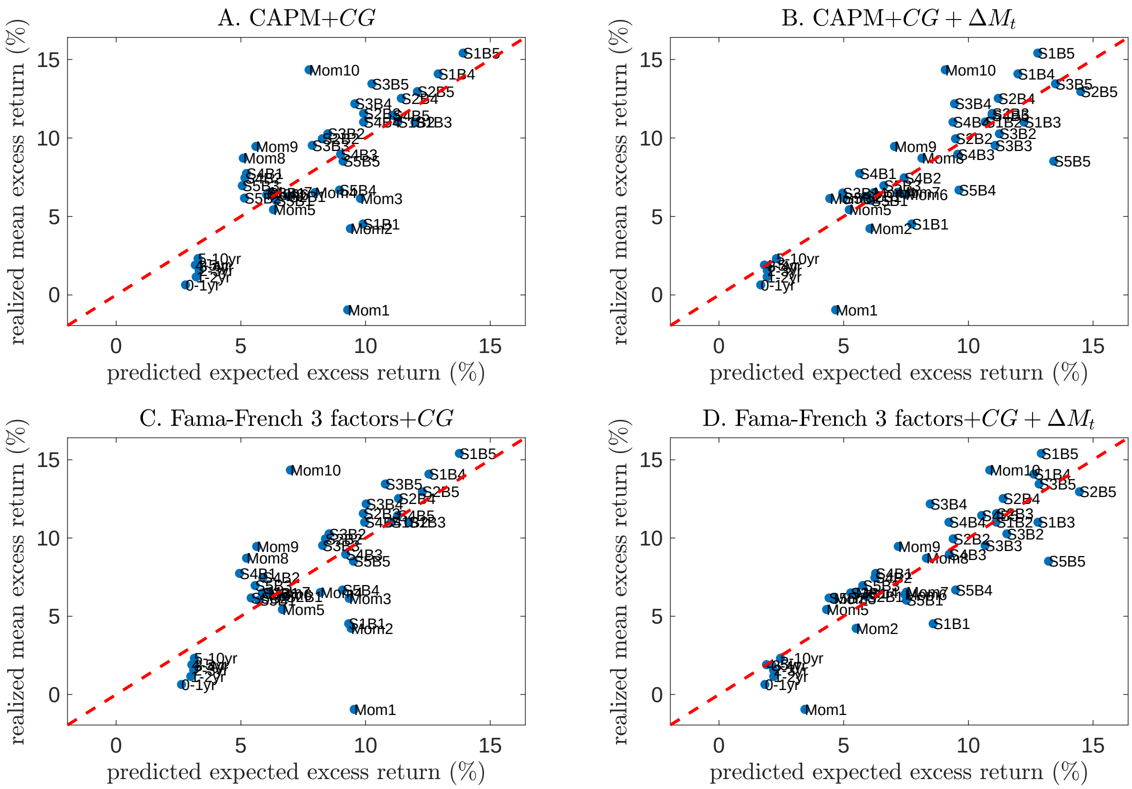
Figure ON.2: Realized versus predicted mean excess returns in factor models with $M_t$.

in columns (7) and (8) of Table ON.II and Panels C and D of Figure ON.3, when we include the misallocation factor $\Delta M_t$ in a four-factor model that contains the Fama-French three factors and accumulated future consumption growth.

The strong pricing power of misallocation factor, as a (macro) nontradable asset pricing factor, is an important, nontrivial empirical finding. As emphasized by Cochrane (2017), it is the sole job of macro-finance to understand what are the primitive sources of systematic risk, by suggesting (macro) nontradable factors, and explain why they earn a premium.[3] However, not many studies find that (macro) nontradable factors motivated by macro-finance models empirically outperform or drive out (ad hoc) tradable factors such as Fama-French factors in explaining the cross section of expected asset returns,[4] partly because the measurement error in nontradable factors causes attenuation bias in the estimates of factor exposures.

[3]Other recent reviews on macro-finance models also highlight this point (e.g., Brunnermeier, Eisenbach and Sannikov, 2012; Dou et al., 2020).

[4]A few exceptions include durable consumption growth (Yogo, 2006; Gomes, Kogan and Yogo, 2009), expenditure shares of housing (Piazzesi, Schneider and Tuzel, 2007), market liquidity (Pástor and Stambaugh, 2003), intermediary leverages (Adrian, Etula and Muir, 2014; He, Kelly and Manela, 2017), and common fund flows (Dou, Kogan and Wu, 2023), among others.




Note: This figure plots the realized mean excess returns of portfolios against the expected excess returns predicted by various linear factor asset pricing models. The sample is yearly and spans the period from 1965 to 2016.

Figure ON.3: Realized versus predicted mean excess returns in factor models with $M_t$ and accumulated future consumption growth.

### 2.2.2 Cash Flow Exposure to the Misallocation Factor

To support the key theoretical mechanism that the slow-moving misallocation drives low-frequency growth fluctuations, we provide further cross-sectional evidence on firms' cash flow exposure to the misallocation factor. Our starting point is the robust evidence found in the asset pricing literature (Bansal, Dittmar and Lundblad, 2005; Parker and Julliard, 2005; Hansen, Heaton and Li, 2008; Santos and Veronesi, 2010): the cash flows of value firms load more positively on accumulated consumption growth than those of growth firms. Given that a higher misallocation predicts a lower consumption growth over long horizons in both the data and model (Panel B of Table III in the main text), if the theoretical mechanism has empirical relevance, we should find the cash flows of value firms load more negatively on misallocation in the data.

To test this prediction, we follow the empirical strategy of Santos and Veronesi (2010). In each year $t$, we sort firms into quintiles based on their book-to-market ratios $BE_{i,t-1}/ME_{i,t-1}$ in year $t-1$, where $BE_{i,t-1}$ is the book equity from Compustat and $ME_{i,t-1}$ is the market equity from CRSP. For each quintile portfolio, we compute the value-weighted return on equity (ROE) across all firms within the portfolio, where a firm's ROE is its income before extraordinary items divided by its common equity. Let $ROE^{p}_{t+j,j+1}$ denote the value-weighted ROE at year $t+j$ of the portfolio $p$, which was formed $j+1$ years earlier, i.e., in year $t-1$. We run a regression similar to the specification

Table ON.III: Exposure to misallocation $M_t$ across firms sorted on the book-to-market ratio.

| $BE_{i,t-1}/ME_{i,t-1}$ | Q1 (low) | Q2 | Q3 | Q4 | Q5 (high) | Q5−Q1 |
|---|---|---|---|---|---|---|
| $\beta_1^p$ | 0.026 | −0.169 | −0.331 | −0.421 | −0.510 | −0.536 |
| | (0.143) | (0.089) | (0.107) | (0.173) | (0.183) | (0.196) |

Note: In each year $t$, we sort firms into quintiles on their book-to-market ratios $BE_{i,t-1}/ME_{i,t-1}$ in year $t-1$. For each quintile portfolio, we estimate $\beta_1^p$ according to specification (ON.29). The sample spans the period from 1965 to 2016. Robust standard errors are reported in brackets.

adopted by Santos and Veronesi (2010), except for including accumulated misallocation shocks as an additional independent variable:

$$\sum_{j=0}^{4} \rho^j ROE^p_{t+j,j+1} = \beta_0^p + \beta_1^p \sum_{j=0}^{4} \rho^j \Delta M_{t+j} + \beta_2^p \sum_{j=0}^{4} \rho^j ROE^{Mkt}_{t+j} + \varepsilon_t, \tag{ON.29}$$

where $\rho = 0.95$ is a constant as in Santos and Veronesi (2010). The variable $\Delta M_t$ is the year-on-year changes in $M_t$ and the variable $ROE_t^{Mkt}$ is the ROE of the market portfolio. The coefficient of interest is $\beta_1^p$, which captures the loadings of accumulated ROE on accumulated misallocation shocks.

Table ON.III presents the results. The accumulated ROE of firms with high book-to-market ratios (i.e., value firms in the quintile group 5 labeled as Q5) is significantly more negatively exposed to accumulated year-on-year changes in misallocation than that of firms with low book-to-market ratios (i.e., growth firms in the quintile group 1 labeled as Q1). The loadings monotonically decrease from 0.026 to −0.510 as the book-to-market ratio increases from Q1 to Q5. The difference in the loadings between Q1 and Q5 (Q5−Q1) is −0.536, which is statistically significant.

### 2.3 Procedure for Nearest Neighbor Matching

For each SIC-3 industry $j$, we calculate the average industry characteristics during the 3-year period before the AJCA (i.e., from 2001 to 2003), $\overline{X}_j = \frac{1}{3}\sum_{t=2001}^{2003} X_{j,t}$, where $X_{j,t}$ is a vector of six industry characteristics, including mean and standard deviation of firms' sales, mean and standard deviation of firms' profit margin, mean and standard deviation of firms' Tobin's Q. We construct a firm's net profit margin using its income before extraordinary items divided by its sales as in Dou, Ji and Wu (2021, 2022), and a firm's Tobin's Q as $Tobin\_Q_{i,t} = (total\_assets_{i,t} + market\_equity_{i,t} - book\_equity_{i,t})/total\_asset_{i,t}$, following Gompers, Ishii and Metrick (2003).

Next, we match each treated industry with an untreated industry which has the shortest Mahalanobis distance from the treated industry. The Mahalanobis distance between any two industries $j$ and $k$ is given by $\sqrt{(\overline{X}_j - \mu)'\Omega^{-1}(\overline{X}_k - \mu)}$, where $\overline{X}_j$ and $\overline{X}_k$ represent the vectors of the six characteristics of industries $j$ and $k$, and $\mu$ and $\Omega$ represent the mean vector and covariance matrix of the six characteristics. This matching process is performed with replacement in untreated industries.

The DID specifications (IA.1) to (IA.4) in the online appendix are estimated with the following weights. Each treated industry is assigned with a weight of 1 and each untreated industry matched to it is also assigned with a weight of 1. Because we allow for replacement, some untreated industries could be matched to multiple treated industries. The weight for such industries is the sum of weights across matches. For example, if an untreated industry is matched with $n$ treated industries, its weight is $n$. If an untreated industry is not matched with any treated industry, its weight is 0.

### 2.4 Impacts of the AJCA on R&D

Complementary to the DID specification in Section 2 in the online appendix, we consider an alternative empirical specification and show that our findings are robust. Specifically, we run the following cross-sectional regression:

$$\Delta\overline{RD}_j = \alpha Treat_j + \beta\overline{X}_j + \varepsilon_j, \tag{ON.30}$$

where the independent variable $\overline{X}_j$ is a vector of average industry-level characteristics over the 3-year period prior to the AJCA, including industry $j$'s mean and standard deviation of firms' sales, mean and standard deviation of firms' profit margin, mean and standard deviation of firms' Tobin's Q. The dependent variable $\Delta\overline{RD}_j$ is the change in industry-level average R&D-capital ratio between the 3-year period prior to the AJCA and the 3-year period after the AJCA, i.e., $\Delta\overline{RD}_j = \frac{1}{3}\sum_{t=2005}^{2007} RD_{j,t} - \frac{1}{3}\sum_{t=2001}^{2003} RD_{j,t}$. The estimated coefficient $\widehat{\alpha}$ in specification (ON.30) is 0.013, with a $p$-value of 0.036, indicating that the AJCA significantly increases the R&D-capital ratio of treated industries relative to untreated industries.

Moreover, we estimate the impact of the AJCA on R&D-capital ratio, controlling for changes in industry-level misallocation by running the following cross-sectional regression:

$$\Delta\overline{RD}_j = \alpha Treat_j + \beta\overline{X}_j + \beta_M\Delta\overline{M}_j + \varepsilon_j, \tag{ON.31}$$

where $\overline{M}_j = \frac{1}{3}\sum_{t=2005}^{2007} M_{j,t} - \frac{1}{3}\sum_{t=2001}^{2003} M_{j,t}$. The estimated coefficient $\widehat{\alpha}$ in specification (ON.31) is 0.013, with a $p$-value of 0.066, suggesting that the AJCA no longer significantly increases the R&D-capital ratio of treated industries relative to untreated industries, after controlling for changes in industry-level misallocation. In other words, our results suggest that the AJCA has positive impacts on treated industries' R&D-capital ratio mainly through the channel of reducing industry-level misallocation.

### 2.5 Estimation Method of Alvarez and Jermann (2004)

We describe the estimation method of Alvarez and Jermann (2004). Alvarez and Jermann (2004) measure the cost of business cycles and the cost of all consumption uncertainty using an approach that does not require the specification of preferences and instead uses asset prices.

#### 2.5.1 Measuring the Costs of Business Cycles and Uncertainty

Let $V_0[\{x\}]$ be the time 0 price of a security that pays $\{x\}$, which is a stochastic process for payoffs for $t \geq 1$. Consider three types of aggregate consumption processes. The process $\{c\}$ represents aggregate consumption; the process $\{\overline{C}\}$ represents aggregate consumption that eliminates all uncertainty, i.e.,

$\overline{C}_t = \mathbb{E}_0[c_t]$; and the process $\{C\}$ represents a moving average of aggregate consumption, given by

$$C_t = \sum_{k=0}^{K} a_k (1+g)^k c_{t-k}, \tag{ON.32}$$

where $g = \mathbb{E}[c_{t+1}/c_t] - 1$ is the unconditional expectation of consumption growth, and the one-sided moving average coefficients $\{a_k\}$ satisfy $\sum_{k=0}^{K} a_k = 1$ and are chosen to represent a low-pass filter that lets pass frequencies that correspond to cycles of eight years and more, which is designed to remove business cycle fluctuations. The initial condition is $c_0/c_{-k} = (1+g)^k$ for $k = 1, ..., K$.

The cost of all uncertainty is defined as the ratio of the value of a claim to the deterministic consumption process $\{\overline{C}\}$ to the value of a claim to the consumption process $\{c\}$:

$$\omega_0^{un} = \frac{V_0[\{\overline{C}\}]}{V_0[\{c\}]} - 1 = \frac{r_0 - g}{y_0 - g} - 1, \tag{ON.33}$$

where $y_0$ and $r_0$ are the yields to maturity that correspond to the prices $V_0[\{\overline{C}\}]$ and $V_0[\{c\}]$, respectively, given by

$$\frac{V_0[\{\overline{C}\}]}{c_0} = \frac{1+g}{y_0 - g} \text{ and } \frac{V_0[\{c\}]}{c_0} = \frac{1+g}{r_0 - g}. \tag{ON.34}$$

The cost of business cycles is defined as the ratio of the value of a claim to the smoothed consumption process $\{C\}$ to the value of a claim to the consumption process $\{c\}$:

$$\omega_0^{bc} = \frac{V_0[\{C\}]}{V_0[\{c\}]} - 1 \approx (r_0 - y) \sum_{k=0}^{K} a_k k, \tag{ON.35}$$

where $\sum_{k=0}^{K} a_k k = 0.387$ as set by Alvarez and Jermann (2004) based on the optimal one-sided filter weights with $K = 20$. The variable $y$ represents the real interest, which is assumed to be a constant. Under this assumption, we have $y = y_0$ because $y_0$ is the yield to maturity of the deterministic process $\{\overline{C}\}$.

### 2.5.2 Valuing Consumption Claims

To estimate the costs of business cycles and uncertainty in equations (ON.33) and (ON.35), a crucial step is to estimate $r_0$, the yield to maturity of the claim to aggregate consumption $\{c\}$. This boils down to estimating the price-dividend ratio of $\{c\}$ according to equation (ON.34). Below, we describe the estimation methods under the assumption of i.i.d. consumption growth. The estimation of the non i.i.d. case follows Appendix C of Alvarez and Jermann (2004). When consumption growth is i.i.d., the price-dividend ratio of $\{c\}$ is a constant, $V_t/c_t \equiv v$. Let $q$ denote the constant price of a security with a single payoff $c'/c = c_{t+1}/c_t$. Then, the price-dividend ratio is given by $v = q/(1-q)$. Alvarez and Jermann (2004) present three methods to estimate $q$.

The first method estimates $q$ by projecting consumption growth onto the payoff space spanned by a set of tradable assets. Consider an observed set of $J+1$ reference assets, which include a risk-free asset. Denote by $\mathbf{R}$ the vector of the real total returns of these assets. Next, we project the consumption payoff $c'/c$ onto the payoff space by estimating the regression $c'/c = \mathbf{b}^T\mathbf{R} + u$, where

$\mathbb{E}[u\mathbf{R}] = 0$. The estimated $q$ is the price of the part of consumption payoff $c'/c$ spanned by $\mathbf{R}$, i.e., $q^* = \mathbf{b}^T\mathbf{1}$, where $\mathbf{1}$ is a vector of 1.

The second method focuses on estimating a lower bound of the price $q$, denoted by $\underline{q}$, based on the $J+1$ reference assets used in the first method. The lower bound of $q$ will provide an upper bound for the estimated costs of business cycles and uncertainty. Specifically, $\underline{q}$ solves

$$\underline{q} = \min_{m \geq 0} \mathbb{E}\left[m\frac{c'}{c}\right], \tag{ON.36}$$

subject to

$$\mathbf{1} = \mathbb{E}[m\mathbf{R}] \text{ and } \frac{\sigma(m)}{\mathbb{E}[m]} \leq h, \tag{ON.37}$$

where $m$ is the SDF that prices all reference assets and limits the Sharpe ratio of any return to be lower than $h$, with $h = 1$. As shown by Cochrane and Saa-Requejo (2000), without imposing the constraint $m \geq 0$, the solution of problem (ON.36) is

$$\underline{q} = q^* - \frac{1}{1+y}\sqrt{h^2 - \tilde{h}^2}\sqrt{1-R^2}\,\sigma\left(\frac{c'}{c}\right), \tag{ON.38}$$

where $R^2$ is the $R$-squared from the regression of $c'/c$ on $\mathbf{R}$ and $\tilde{h}$ is the highest Sharpe ratio achievable with the reference assets

The third method estimates $q$ based on a parametric model for the SDF. In particular, $m_{t+1}$ is specified as follows

$$m_{t+1} = \delta\exp(\lambda^T\mathbf{f}_{t+1}), \tag{ON.39}$$

where $\mathbf{f}_{t+1}$ is a vector of factors with loading vector $\lambda$ and $\delta$ is a constant. Using the reference assets, factor loadings $\lambda$ are estimated by generalized method of moments on

$$\mathbf{0} = \mathbb{E}\left[\exp(\lambda^T\mathbf{f}_{t+1})\left(\mathbf{R}_{t+1} - (1+y)\right)\right]. \tag{ON.40}$$

Under the assumption that the factors and the returns are i.i.d., we estimate $q$ through the sample analogue to

$$0 = \mathbb{E}\left[\exp(\lambda^T\mathbf{f}_{t+1})\left(\frac{c_{t+1}/c_t}{q} - (1+y)\right)\right]. \tag{ON.41}$$

### 2.5.3 Results

The implementation details of all three methods closely follow Alvarez and Jermann (2004). We use the average real annual yield for long-term government bonds from the Federal Reserve Economic Database to measure $y$, which is equal to $y_0$ under the assumption of a constant real interest rate. The aggregate annual consumption is measured by per-capita real personal consumption expenditures on nondurable goods and services.

Table ON.IV presents the estimates for the costs of business cycles and uncertainty for different specifications. Panels A and B present the estimates of the first and second methods, respectively. We consider three sets of reference assets. In addition to a risk-free rate, the three sets include the returns

Table ON.IV: Marginal cost of consumption fluctuations: 1965-2016.

| | Business cycle frequency (%) | | All uncertainty (%) | |
|---|---|---|---|---|
| Panel A: Method 1 | | | | |
| | i.i.d. | non i.i.d. | i.i.d. | non i.i.d. |
| Reference assets: R(Market) | 0.05 | 0.08 | 21.82 | 33.77 |
| Reference assets: R(10dec) | 0.06 | 0.29 | 23.97 | 117.09 |
| Reference assets: R(17ind) | 0.01 | 0.16 | 6.03 | 65.24 |
| Panel B: Method 2 (i.e., estimation of upper bound) | | | | |
| | i.i.d. | non i.i.d. | i.i.d. | non i.i.d. |
| Reference assets: R(Market) | 0.53 | 0.84 | 213.08 | 338.12 |
| Reference assets: R(10dec) | 0.49 | 0.86 | 198.64 | 346.39 |
| Reference assets: R(17ind) | 0.40 | 0.37 | 160.92 | 150.04 |
| Panel C: Method 3 | | | | |
| Factors: $\Delta \ln c_t$ | 0.72 | | 291.78 | |
| Factors: $\Delta \ln c_t$, R(Market) | 0.47 | | 188.78 | |

Note: R(Market) stands for the CRSP value-weighted return covering NYSE and AMEX; R(10dec) stands for the returns of the 10 CRSP size-decile portfolios; R(17ind) stands for the returns of the 17 industry portfolios from Kenneth R. French Data Library. All returns are real. $\Delta \ln c_t$ stands for consumption growth. The sample is yearly and spans the period from 1965 to 2016.

of the Center for Research in Security Prices (CRSP) value-weighted portfolio covering the New York Stock Exchange (NYSE) and the American Stock Exchange (AMEX), 10 CRSP size-decile portfolios, and 17 industry portfolios from the Kenneth R. French Data Library, respectively. For each set of reference assets, we do the estimation for both the specification of i.i.d. consumption growth and the specification in which consumption growth is captured by a two-state Markov regime-switching process. Regimes are determined by splitting the sample into high and low consumption growth. We set the cutoff at 0.5% below the average consumption growth rate in the sample to capture the difference between recessions and expansions. Within each regime, consumption growth is i.i.d. Panel C presents the estimates of the third method. We consider two sets of factors. In one set, we use the log consumption growth rate as the only factor and choose $\lambda$ to fit the market return. In the other set, we additionally include the log market return as a second factor and choose $\lambda$ to fit the market return and the return difference between the smallest and largest CRSP size-decile portfolios.

## 2.6 Estimation Method of Eisfeldt and Shi (2018)

Following the method of Eisfeldt and Shi (2018), we compute the average output gain in recessions if the amount of capital reallocation observed in booms could be achieved. The estimation is conducted using the sample for constructing the empirical misallocation measure in Section 4.1 in the main text; it takes several steps, which we describe below.

### 2.6.1 Estimating Production Function

Because our goal is to get a benchmark estimate that can be compared to our model-implied estimate for validation purposes, we estimate a production function similar to that estimated by Eisfeldt and Shi (2018). This production function is different from the one specified in our model due to the absence of intermediate goods. Specifically, following Eisfeldt and Shi (2018), we assume that firm $i$ uses a Cobb-Douglas production function to produce output $Y_{i,t}$ using capital $K_{i,t}$ and labor $L_{i,t}$.

$$Y_{i,t} = z_{i,t} K_{i,t}^{\alpha} L_{i,t}^{\beta_i}, \tag{ON.42}$$

where the parameter $\alpha$ is the capital share, assumed to be identical across firms, $\beta_i$ is the labor share, and $z_{i,t}$ is the firm's productivity. In the data, we measure firm $i$'s output $Y_{i,t}$ and capital $K_{i,t}$ in year $t$ using its sales, $sale_{i,t}$, and net property, plant and equipment, $ppent_{i,t}$, respectively, as in Eisfeldt and Shi (2018). We measure firm $i$'s labor $L_{i,t}$ using its number of employees, $emp_{i,t}$, as in the literature (e.g., Griffith, Harrison and Reenen, 2006).

The parameters $\alpha$ and $\beta_i$ are estimated to minimize the total squared error of the residuals, $\sum_i \sum_t \varepsilon_{i,t}^2$, where $\varepsilon_{i,t}$ is the residual from the time series regression for firm $i$,

$$\ln Y_{i,t} - \alpha \ln K_{i,t} = c_i + \phi_i d_t + \beta_i \ln L_{i,t} + \varepsilon_{i,t}. \tag{ON.43}$$

The dummy variable $d_t$ is an indicator for whether year $t$ is a boom or recession year. Booms and recessions are defined as years in which per-capita real GDP is above or below its HP filtered trend, respectively. As in Eisfeldt and Shi (2018), firms with $\widehat{\beta}_i$ less than 0 or greater than 1 are dropped to ensure that the production function has decreasing returns to scale in labor. The productivity $z_{i,t}$ for each firm-year is estimated by

$$\widehat{z}_{i,t} = Y_{i,t} / (K_{i,t}^{\widehat{\alpha}} L_{i,t}^{\widehat{\beta}_i}). \tag{ON.44}$$

### 2.6.2 Reallocation of Capital

In the data, the observed aggregate capital reallocation rate in year $t$ is calculated by

$$R_t = \frac{\sum_i sppe_{i,t} + aqc_{i,t}}{\sum_i at_{i,t-1}}. \tag{ON.45}$$

The variables $sppe_{i,t}$ and $aqc_{i,t}$ are firm $i$'s sale of property and acquisitions in year $t$, which measure capital sales and purchases by firm $i$, respectively. The variable $at_{i,t-1}$ is firm $i$'s total assets in year $t-1$. We compute the average rate of capital reallocation in booms and recessions, denoted by $\overline{R}^b$ and $\overline{R}^r$, as the simple average of $R_t$ over all boom years and recessions years in our sample, respectively. We find that $\overline{R}^b > \overline{R}^r$, consistent with the finding of Eisfeldt and Shi (2018) that capital reallocation is procyclical.

To gauge the average output gain in recessions if the amount of capital reallocation observed in booms could be achieved, we perform two counterfactual experiments for capital reallocation in all recession years in our sample.

In the first counterfactual experiment, we efficiently reallocate the incremental amount of capital in each recession year $t$ based on the capital stock in year $t-1$ to maximize the aggregate output in year $t$, subject to the observed aggregate capital reallocation rate $R_t$. Specifically, in each recession year $t$, we solve the following problem:

$$Y_t(R_t) = \max_{\{\widetilde{K}_{i,t}\}_i} \quad \sum_i \widehat{z}_{i,t} \widetilde{K}_{i,t}^{\widehat{\alpha}} L_{i,t}^{\widehat{\beta}_i}, \tag{ON.46}$$

subject to

$$\frac{\sum_i |\widetilde{K}_{i,t} - K_{i,t-1}|}{\sum_i at_{i,t-1}} = R_t, \tag{ON.47}$$

$$\sum_i (\widetilde{K}_{i,t} - K_{i,t-1}) = 0. \tag{ON.48}$$

Intuitively, the constraint (ON.47) restricts the reallocation rate in year $t$ to be the same as the one calculated by equation (ON.45) in the data. The constraint (ON.48) restricts the total purchase of capital to be the same as the total sales of capital. The optimal solution of $\widetilde{K}_{i,t}$ can be obtained by a simple numerical algorithm, which essentially reallocates the capital of firms with lowest marginal product of capital (MPK) to those with the highest MPK. We describe the algorithm at the end of this section.

In the second counterfactual experiment, we efficiently reallocate the incremental amount of capital in each recession year $t$ to maximize the aggregate output in year $t$, subject to the average capital reallocation rate in booms. The constrained maximization problem is similar to (ON.46) except for replacing $R_t$ in the constraint (ON.47) with $\overline{R}^b$. This constraint now becomes looser because $\overline{R}^b$ is usually greater than $R_t$ in recession years. Let $Y_t(\overline{R}^b)$ denote the aggregate output in recession year $t$ in the second counterfactual experiment.

Below, we describe the algorithm that solves problem (ON.46) in each recession year $t$.

(i) Based on the estimated production function, we compute the ex-ante MPK as

$$MPK_{i,t} = \widehat{z}_{i,t} \widehat{\alpha} K_{i,t-1}^{\widehat{\alpha}-1} L_{i,t}^{\widehat{\beta}_i}.$$

To alleviate the effect of outliers, we follow Eisfeldt and Shi (2018) and drop firms whose $MPK_{i,t}$ are in the top and bottom 1%.

(ii) Sort firms by $MPK_{i,t}$ in descending order. Consider two cutoff productivities $MPK_t^b$ and $MPK_t^s$, satisfying

$$\min_i(MPK_{i,t}) < MPK_t^s < MPK_t^b < \max_i(MPK_{i,t}). \tag{ON.49}$$

Capital is reallocated according to the following rule

(a) Firms whose $MPK_{i,t} > MPK_t^b$ buy an amount of capital equal to $\widetilde{K}_{i,t} - K_{i,t-1}$, where $\widetilde{K}_{i,t} = \left( \frac{MPK_t^b}{\widehat{z}_{i,t} \widehat{\alpha} L_{i,t}^{\widehat{\beta}_i}} \right)^{1/(\widehat{\alpha}-1)}$.

(b) Firms whose $MPK_{i,t} < MPK_t^s$ sell an amount of capital equal to $K_{i,t-1} - \widetilde{K}_{i,t}$, where

$$\widetilde{K}_{i,t} = \left( \frac{MPK_t^s}{\widehat{z}_{i,t}\widehat{\alpha} L_{i,t}^{\widehat{\beta}_i}} \right)^{1/(\widehat{\alpha}-1)}.$$

(c) The other firms whose $MPK_{i,t} \in [MPK_t^s, MPK_t^b]$ do not buy or sell, i.e., $\widetilde{K}_{i,t} = K_{i,t-1}$.

The above rule ensures that, after reallocation, the MPK of all capital buyers equals $MPK_t^b$, the MPK of all capital sellers equals $MPK_t^s$, and the MPK of all other firms that do not buy or sell is between $MPK_t^s$ and $MPK_t^b$. The values of cutoff productivities $MPK_t^b$ and $MPK_t^s$ are chosen to satisfy the constraints (ON.47) and (ON.48), meaning

$$\frac{\sum_{i \in \text{buyers}} |\widetilde{K}_{i,t} - K_{i,t-1}|}{\sum_i at_{i,t-1}} = \frac{\sum_{i \in \text{sellers}} |\widetilde{K}_{i,t} - K_{i,t-1}|}{\sum_i at_{i,t-1}} = \frac{1}{2} R_t. \tag{ON.50}$$

We provide a proof for the optimality of the rule of reallocation.

*Proof.* First, we show that all capital buyers (and sellers) should have the same MPK after capital reallocation. Suppose in optimum, there exist buyers $i \neq j$ with $MPK_{i,t} > MPK_{j,t}$. Then due to the continuity of the output function $Y_{i,t}(K) = \widehat{z}_{i,t} K^{\widehat{\alpha}} L_{i,t}^{\widehat{\beta}_i}$, there exists a small enough $\Delta > 0$, such that $Y_{i,t}(\widetilde{K}_{i,t} + \Delta) + Y_{j,t}(\widetilde{K}_{j,t} - \Delta) - Y_{i,t}(\widetilde{K}_{i,t}) - Y_{j,t}(\widetilde{K}_{j,t}) \geq \frac{\Delta}{2}(MPK_{i,t} - MPK_{j,t}) > 0$. So moving $\Delta$ capital from $j$ to $i$ increases the aggregate output while still satisfying the constraints. This means in optimum, all buyers' MPK must be the same and equal to $MPK_t^b$. Similarly, all sellers' MPK must be equal to $MPK_t^s$.

Second, we show that all firms whose $MPK_{i,t} > MPK_t^b$ must be capital buyers, and those whose $MPK_{i,t} < MPK_t^s$ must be capital sellers. Suppose in optimum, there exists firm $i$ who does not buy capital but its $MPK_{i,t} > MPK_t^b$. Let $j$ be one of firms that buys capital. According to the proof above, it must be the case that $MPK_{j,t} = MPK_t^b$. Then, there exists a small enough $\Delta > 0$, such that $Y_{i,t}(\widetilde{K}_{i,t} + \Delta) + Y_{j,t}(\widetilde{K}_{j,t} - \Delta) - Y_{i,t}(\widetilde{K}_{i,t}) - Y_{j,t}(\widetilde{K}_{j,t}) \geq \frac{\Delta}{2}(MPK_{i,t} - MPK_t^b) > 0$. So moving $\Delta$ capital from $j$ to $i$ increases the aggregate output while satisfying the constraints. Thus, all firms with $MPK_{i,t} > MPK_t^b$ should buy capital. Similarly, all firms with $MPK_{i,t} < MPK_t^s$ should sell capital and the other firms whose $MPK_{i,t} \in [MPK_t^s, MPK_t^b]$ do not buy or sell.

□

## 3 Numerical Algorithm

Our model can be solved either using a local perturbation approach or a global approach based on value function iterations. Because the aggregate dynamics do not feature occasionally binding constraints or region-dependent policy rules, the local perturbation approach can be easily implemented in dynare. Here, we present the numerical algorithm for the global approach based on value function iterations.

We discretize the model with time interval $\Delta t$. The Brownian motion shock $\mathrm{d}W_t$ takes two values, $\sqrt{\Delta t}$ and $-\sqrt{\Delta t}$, with equal probabilities. Define $\Gamma_t \equiv \mathrm{Cov}(\widetilde{a}_{i,t}, \widetilde{z}_{i,t}) = -M_t \mathrm{var}(\widetilde{z}_{i,t}) = -M_t \sigma^2/2$. The economy is summarized by the evolution of two endogenous state variables, $E_t \equiv N_t / A_t$ and $\Gamma_t$.

We use superscripts $+$ and $-$ to denote variables at $t+\Delta t$, corresponding to $\mathrm{d}W_t = \sqrt{\Delta t}$ and $\mathrm{d}W_t = -\sqrt{\Delta t}$, respectively. The endogenous state variable $\Gamma_t$ evolves according to equation (IA.55) in the online appendix:

$$\Gamma_{t+\Delta t} = \Gamma_t - \theta\Gamma_t\Delta t + \mathrm{Cov}(\widetilde{z}_{i,t}, \Delta\widetilde{a}_{i,t}), \tag{ON.51}$$

where $\mathrm{Cov}(\widetilde{z}_{i,t}, \Delta\widetilde{a}_{i,t})$ is given by equation (IA.63) in the online appendix, as follows:

$$\begin{aligned}\mathrm{Cov}(\widetilde{z}_{i,t}, \Delta\widetilde{a}_{i,t}) &= \frac{(1+\lambda)\sigma^2\kappa_t}{2}\exp\left(\frac{\sigma^2}{4}\right)\Phi\left(\frac{\sigma^2/2-\widetilde{\underline{z}}_t}{\sigma/\sqrt{2}}\right)\Delta t \\ &+ \frac{(1+\lambda)\sigma}{2\sqrt{\pi}}\left[(\underline{z}_t\kappa_t - r_{f,t} - \delta_k - 0.5(1+\lambda)\sigma_k^2 + \sigma_k\sigma_{a,t})\Delta t - \sigma_k \mathrm{d}W_t\right]\exp\left(-\frac{\widetilde{\underline{z}}_t^2}{\sigma^2}\right).\end{aligned} \tag{ON.52}$$

Let $\Gamma^+_{t+\Delta t}$ and $\Gamma^-_{t+\Delta t}$ be the value of $\Gamma_{t+\Delta t}$ corresponding to $\mathrm{d}W_t = \sqrt{\Delta t}$ and $\mathrm{d}W_t = -\sqrt{\Delta t}$, respectively. In equation (ON.52), the variables $\kappa_t$, $\widetilde{\underline{z}}_t$, and $r_{f,t}$ are given by equations (IA.36) and (IA.19) in the online appendix, and the SDF, respectively, as follows:

$$\kappa_t = \alpha(1-\varepsilon)H_t^{-\frac{1}{\alpha}}\frac{Y_t}{A_t}\frac{A_t}{K_t}, \tag{ON.53}$$

$$\underline{z}_t\kappa_t = r_{f,t} + \delta_k + \sigma_k(\sigma_{\xi,t}(\underline{z}_t) - \eta_t), \tag{ON.54}$$

$$r_{f,t} = -\frac{1}{\Delta t}\ln\left(\mathbb{E}_t\left[\frac{\Lambda_{t+\Delta t}}{\Lambda_t}\right]\right), \tag{ON.55}$$

where $Y_t/A_t$, $H_t$, and $K_t/A_t$ are functions of state variables $E_t$ and $\Gamma_t$, given by equations (IA.34), (IA.50), and (IA.48) in the online appendix, respectively, as follows:

$$\frac{Y_t}{A_t} = (\varepsilon\nu)^{\frac{\varepsilon}{1-\varepsilon}}H_tE_t^{1-\alpha}\left(\frac{K_t}{A_t}\right)^{\alpha}, \tag{ON.56}$$

$$H_t = \left[(1+\lambda)\frac{A_t}{K_t}\exp\left(\Gamma_t + \frac{\sigma^2}{4}\right)\Phi\left(\Phi^{-1}\left(\frac{1}{1+\lambda}\frac{K_t}{A_t}\right) + \frac{\sigma}{\sqrt{2}}\right)\right]^{\alpha}, \tag{ON.57}$$

$$\widetilde{\underline{z}}_t = \Gamma_t - \Phi^{-1}\left(\frac{1}{1+\lambda}\frac{K_t}{A_t}\right)\frac{\sigma}{\sqrt{2}}. \tag{ON.58}$$

The endogenous state variable $E_t$ evolves according to

$$\frac{\Delta E_t}{E_t} = \frac{\Delta N_t}{N_t} - \frac{\Delta A_t}{A_t}. \tag{ON.59}$$

Substituting equations (36) and (37) in the main text into the above equation, we obtain

$$\begin{aligned}\frac{E_{t+\Delta t}}{E_t} =& 1 + \chi\left(\chi q_t\right)^{\frac{1-h}{h}}\Delta t - \alpha(1-\varepsilon)\frac{Y_t}{A_t}\Delta t + (r_{f,t}+\delta_k)\frac{K_t}{A_t}\Delta t \\ &+ (\rho + \delta_a - \delta_b - r_{f,t})\Delta t.\end{aligned} \tag{ON.60}$$

Let $E^+_{t+\Delta t}$ and $E^-_{t+\Delta t}$ be the value of $E_{t+\Delta t}$ corresponding to $\mathrm{d}W_t = \sqrt{\Delta t}$ and $\mathrm{d}W_t = -\sqrt{\Delta t}$, respectively.

In equation (ON.60), the variable $q_t = q(E_t, \Gamma_t)$ is given by equation (6) in the main text; it is a function of state variables $(E_t, \Gamma_t)$ and can be solved recursively as follows

$$\begin{aligned} q(E_t, \Gamma_t) =& \frac{1}{1+\delta_b \Delta t} \left( \pi_t \Delta t + \mathbb{E}_t \left[ \frac{\Lambda_{t+\Delta t}}{\Lambda_t} q(E_{t+\Delta t}, \Gamma_{t+\Delta t}) \right] \right) \\ =& \frac{1}{1+\delta_b \Delta t} \left( \pi_t \Delta t + \frac{1}{2} \frac{\Lambda^+_{t+\Delta t}}{\Lambda_t} q(E^+_{t+\Delta t}, \Gamma^+_{t+\Delta t}) + \frac{1}{2} \frac{\Lambda^-_{t+\Delta t}}{\Lambda_t} q(E^-_{t+\Delta t}, \Gamma^-_{t+\Delta t}) \right], \end{aligned} \tag{ON.61}$$

where $\pi_t$ is given by equation (IA.38) in the online appendix:

$$\pi_t = \frac{(1-\nu)\varepsilon}{E_t} \frac{Y_t}{A_t}. \tag{ON.62}$$

Epstein and Zin (1989) show that the SDF in equation (12) in the main text is equivalent to

$$\frac{\Lambda_{t+\Delta t}}{\Lambda_t} = e^{-\frac{\delta(1-\gamma)}{1-1/\psi}\Delta t} \left( \frac{C_{t+\Delta t}}{C_t} \right)^{-\frac{1-\gamma}{\psi(1-1/\psi)}} \left(1 + R_{m,t+\Delta t}\Delta t\right)^{\frac{1/\psi - \gamma}{1-1/\psi}}, \tag{ON.63}$$

where $R_{m,t+\Delta t}$ is the net return on wealth (or the consumption claim's return)

$$1 + R_{m,t+\Delta t}\Delta t = \frac{W_{t+\Delta t}}{W_t - C_t \Delta t}. \tag{ON.64}$$

We have

$$\mathbb{E}_t \left[ \frac{\Lambda_{t+\Delta t}}{\Lambda_t} (1 + R_{m,t+\Delta}\Delta t) \right] = 1. \tag{ON.65}$$

Substituting equations (ON.63) and (ON.64) into (ON.65), we obtain

$$1 = \mathbb{E}_t \left[ e^{-\frac{\delta(1-\gamma)}{1-1/\psi}\Delta t} \left( \frac{C_{t+\Delta t}}{C_t} \right)^{-\frac{1-\gamma}{\psi - 1}} \left( \frac{W_{t+\Delta t}}{C_{t+\Delta t}} \frac{C_{t+\Delta t}}{C_t} \frac{1}{W_t/C_t - \Delta t} \right)^{\frac{1-\gamma}{1-1/\psi}} \right]. \tag{ON.66}$$

Rearranging the above equation, we obtain

$$\frac{W_t}{C_t} = \Delta t + e^{-\delta \Delta t} \mathbb{E}_t \left[ \left( \frac{C_{t+\Delta t}}{C_t} \right)^{1-\gamma} \left( \frac{W_{t+\Delta t}}{C_{t+\Delta t}} \right)^{\frac{1-\gamma}{1-1/\psi}} \right]^{\frac{1-1/\psi}{1-\gamma}}. \tag{ON.67}$$

The wealth-consumption ratio $W_t/C_t$ is a function of state variables, denoted by $WC_t \equiv WC(E_t, \Gamma_t)$. Let $C^+_{t+\Delta t}$ and $C^-_{t+\Delta t}$ be the value of $C_{t+\Delta t}$ corresponding to $\mathrm{d}W_t = \sqrt{\Delta t}$ and $\mathrm{d}W_t = -\sqrt{\Delta t}$, respectively. We can rewrite equation (ON.67) as

$$WC_t = \Delta t + e^{-\delta \Delta t} \left[ \frac{1}{2} \left( \frac{C^+_{t+\Delta t}}{C_t} \right)^{1-\gamma} (WC^+_{t+\Delta t})^{\frac{1-\gamma}{1-1/\psi}} + \frac{1}{2} \left( \frac{C^-_{t+\Delta t}}{C_t} \right)^{1-\gamma} (WC^-_{t+\Delta t})^{\frac{1-\gamma}{1-1/\psi}} \right]^{\frac{1-1/\psi}{1-\gamma}}, \tag{ON.68}$$

where

$$WC^+_{t+\Delta t} = WC(E^+_{t+\Delta t}, \Gamma^+_{t+\Delta t}), \tag{ON.69}$$

$$WC^{-}_{t+\Delta t} = WC(E^{-}_{t+\Delta t}, \Gamma^{-}_{t+\Delta t}). \tag{ON.70}$$

The aggregate consumption is given by the budget constraint (ON.18):

$$\begin{aligned} \frac{C_t}{A_t} =& \frac{w_t}{A_t} + \frac{D_t}{A_t} + r_{f,t}\frac{B_t}{A_t} - \left(\frac{B_{t+\Delta t}}{A_{t+\Delta t}}\frac{A_{t+\Delta t}}{A_t} - \frac{B_t}{A_t}\right)\frac{1}{\Delta t} \\ =& \frac{w_t}{A_t} + \frac{D_t}{A_t} + r_{f,t}\left(\frac{K_t}{A_t} - 1\right) - \left[\left(\frac{K_{t+\Delta t}}{A_{t+\Delta t}} - 1\right)\frac{A_{t+\Delta t}}{A_t} - \left(\frac{K_t}{A_t} - 1\right)\right]\frac{1}{\Delta t}. \end{aligned} \tag{ON.71}$$

Because $C_t$ is known (i.e., $\mathrm{d}B_t/B_t$ is locally deterministic), theoretically we have

$$\left(\frac{K^{+}_{t+\Delta t}}{A^{+}_{t+\Delta t}} - 1\right)\frac{A^{+}_{t+\Delta t}}{A_t} = \left(\frac{K^{-}_{t+\Delta t}}{A^{-}_{t+\Delta t}} - 1\right)\frac{A^{-}_{t+\Delta t}}{A_t}, \tag{ON.72}$$

where $K^{+}_{t+\Delta t}$, $A^{+}_{t+\Delta t}$ and $K^{-}_{t+\Delta t}$, $A^{-}_{t+\Delta t}$ are the values of $K_{t+\Delta t}$, $A_{t+\Delta t}$ corresponding to $\mathrm{d}W_t = \sqrt{\Delta t}$ and $\mathrm{d}W_t = -\sqrt{\Delta t}$, respectively. Because of property (ON.72), the numerical error caused by discretization is minimized by using $0.5\left(\frac{K^{+}_{t+\Delta t}}{A^{+}_{t+\Delta t}} - 1\right)\frac{A^{+}_{t+\Delta t}}{A_t} + 0.5\left(\frac{K^{-}_{t+\Delta t}}{A^{-}_{t+\Delta t}} - 1\right)\frac{A^{-}_{t+\Delta t}}{A_t}$ to approximate $\left(\frac{K_{t+\Delta t}}{A_{t+\Delta t}} - 1\right)\frac{A_{t+\Delta t}}{A_t}$ in equation (ON.71). Thus, the term $C_t/A_t \equiv CA(E_t, \Gamma_t)$ in equation (ON.71) can be solved as a function of state variables $E_t$ and $\Gamma_t$.

The consumption growth terms in equation (ON.68) are given by

$$\frac{C^{+}_{t+\Delta t}}{C_t} = \frac{CA(E^{+}_{t+\Delta t}, \Gamma^{+}_{t+\Delta t})}{CA(E_t, \Gamma_t)}\frac{A_{t+\Delta t}}{A_t}, \tag{ON.73}$$

$$\frac{C^{-}_{t+\Delta t}}{C_t} = \frac{CA(E^{-}_{t+\Delta t}, \Gamma^{-}_{t+\Delta t})}{CA(E_t, \Gamma_t)}\frac{A_{t+\Delta t}}{A_t}. \tag{ON.74}$$

The variables $w_t/A_t$ and $D_t/A_t$ are given by (30) in the main text and (ON.13):

$$\frac{w_t}{A_t} \equiv wA(E_t, \Gamma_t) = (1-\alpha)(1-\varepsilon)\frac{Y_t}{A_t}, \tag{ON.75}$$
$$\frac{D_t}{A_t} \equiv DA(E_t, \Gamma_t) = \rho + (1-\nu)\varepsilon\frac{Y_t}{A_t} - \frac{S_t}{A_t}, \tag{ON.76}$$

where $S_t/A_t$ is given by equation (31) in the main text:

$$\frac{S_t}{A_t} = \frac{S_t}{N_t}E_t = \left(\chi q(E_t, \Gamma_t)\right)^{\frac{1}{h}} E_t. \tag{ON.77}$$

The variables $A_{t+\Delta t}/A_t$ is given by equation (36) in the main text:

$$\frac{A_{t+\Delta t}}{A_t} = 1 + \alpha(1-\varepsilon)\frac{Y_t}{A_t}\Delta t - (r_{f,t} + \delta_k)\frac{K_t}{A_t}\Delta t - (\rho + \delta_a - r_{f,t})\Delta t. \tag{ON.78}$$

After solving the $WC(E_t, \Gamma_t)$ ratio from equation (ON.68), substituting into the equation (ON.63) to obtain the SDF:

$$\frac{\Lambda^{+}_{t+\Delta t}}{\Lambda_t} = e^{-\frac{\delta(1-\gamma)}{1-1/\psi}\Delta t}\left(\frac{C^{+}_{t+\Delta t}}{C_t}\right)^{-\gamma}\left(\frac{WC(E^{+}_{t+\Delta t}, \Gamma^{+}_{t+\Delta t})}{WC(E_t, \Gamma_t) - \Delta t}\right)^{\frac{1/\psi-\gamma}{1-1/\psi}}, \tag{ON.79}$$

$$\frac{\Lambda^-_{t+\Delta t}}{\Lambda_t} = e^{-\frac{\delta(1-\gamma)}{1-1/\psi}\Delta t}\left(\frac{C^-_{t+\Delta t}}{C_t}\right)^{-\gamma}\left(\frac{WC(E^-_{t+\Delta t},\Gamma^-_{t+\Delta t})}{WC(E_t,\Gamma_t)-\Delta t}\right)^{\frac{1/\psi-\gamma}{1-1/\psi}}. \tag{ON.80}$$

**Welfare.** In discrete time, the preference specified in equation (10) in the main text is

$$U_t = \left[(1-e^{-\delta\Delta t})C_t^{1-1/\psi} + e^{-\delta\Delta t}\left(\mathbb{E}_t\left[(U_{t+\Delta t})^{1-\gamma}\right]\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}. \tag{ON.81}$$

Dividing both sides by $C_t$,

$$\left(\frac{U_t}{C_t}\right)^{1-1/\psi} = (1-e^{-\delta\Delta t}) + e^{-\delta\Delta t}\left(\mathbb{E}_t\left[\left(\frac{C_{t+\Delta t}}{C_t}\frac{U_{t+\Delta t}}{C_{t+\Delta t}}\right)^{1-\gamma}\right]\right)^{\frac{1-1/\psi}{1-\gamma}}. \tag{ON.82}$$

**Steps of Implementing the Numerical Algorithm.** Following the standard practice, we discretize the state variables $(E_t, \Gamma_t)$ into dense grids. The values that do not fall on any grid are obtained by linear interpolation or extrapolation. We then solve the model in the steps listed below. Because we need to solve a large number of nonlinear equations, we use the commercial nonlinear solver *knitro*.[5] All the programs are written in C++ with parallel computing in a state-of-the-art server of 56 cores.

(1) Guess $q(E_t, \Gamma_t) = 0.1$ for all states.

(2) Guess $\sigma_{\xi}(\underline{z}_t, E_t, \Gamma_t) = 0$ for all states.

(3) Guess $\eta(E_t, \Gamma_t) = 0$ for all states.

(4) Solve the evolution of endogenous state variables $E_t$ and $\Gamma_t$.

(5) Solve equation (ON.68) using *knitro* to obtain the wealth-consumption ratio as a function of state variables, i.e., $WC(E_t, \Gamma_t)$.

(6) Solve equations (ON.79) and (ON.80) to obtain the SDF as a function of state variables, i.e.,

$$\frac{\Lambda^+_{t+\Delta t}}{\Lambda_t} \equiv SDF(E^+_{t+\Delta t}, \Gamma^+_{t+\Delta t}), \tag{ON.83}$$

$$\frac{\Lambda^-_{t+\Delta t}}{\Lambda_t} \equiv SDF(E^-_{t+\Delta t}, \Gamma^-_{t+\Delta t}). \tag{ON.84}$$

Next, calculate the market price of risk $\eta_t$ in equation (IA.14) in the online appendix as follows

$$\widehat{\eta}(E_t, \Gamma_t) = -\frac{SDF(E^+_{t+\Delta t}, \Gamma^+_{t+\Delta t}) - SDF(E^-_{t+\Delta t}, \Gamma^-_{t+\Delta t})}{2\sqrt{\Delta t}}. \tag{ON.85}$$

If $\max|\widehat{\eta}(E_t, \Gamma_t) - \eta(E_t, \Gamma_t)| < 10^{-9}$, stop. Otherwise, jump to step (4) using $\widehat{\eta}(E_t, \Gamma_t)$ as the initial guess of $\eta(E_t, \Gamma_t)$.

[5] See https://www.artelys.com/solvers/knitro for more details.

(7) Solve managers' problem in equation (15) in the main text to obtain $\sigma_\xi(\underline{z}_t, E_t, \Gamma_t)$. This is achieved in the following substeps.

(7.1) Problem (15) in the main text can be simplified because it is linear in $a_{i,t}$ (see equation (IA.13) in the online appendix). This means that we only need to solve $\xi(z_{i,t}, E_t, \Gamma_t)$ recursively as follows

$$\xi(z_{i,t}, E_t, \Gamma_t) = \tau\Delta t + \mathbb{E}_t\left[\frac{\Lambda_{t+\Delta t}}{\Lambda_t}\frac{a_{i,t+\Delta t}}{a_{i,t}}\xi(z_{i,t+\Delta t}, E_{t+\Delta t}, \Gamma_{t+\Delta t})\right]. \tag{ON.86}$$

The evolution $a_{i,t+\Delta t}/a_{i,t}$ is given by equations (2) and (21) in the main text:

$$\begin{aligned}\frac{a_{i,t+\Delta t}}{a_{i,t}} =& 1 + (1+\lambda)\left(\kappa_t z_{i,t}\mathrm{d}t - \delta_k\mathrm{d}t - \sigma_k\mathrm{d}W_t - r_{f,t}\mathrm{d}t\right)\mathbb{1}_{z_{i,t}\geq \underline{z}_t}\\ &+ (r_{f,t} - \rho - \delta_a)\mathrm{d}t + \sigma_{a,t}\mathrm{d}W_t.\end{aligned} \tag{ON.87}$$

Substituting equation (ON.87) into (ON.86), we obtain

$$\begin{aligned}\xi_{i,t} =& \tau\Delta t + \mathbb{E}_t\left[\frac{\Lambda_{t+\Delta t}}{\Lambda_t}\left[1 + (1+\lambda)\left(\kappa_t z_{i,t}\mathrm{d}t - \delta_k\mathrm{d}t - \sigma_k\mathrm{d}W_t - r_{f,t}\mathrm{d}t\right)\mathbb{1}_{z_{i,t}\geq \underline{z}_t}\right]\xi_{i,t+\Delta t}\right]\\ &+ \mathbb{E}_t\left[\frac{\Lambda_{t+\Delta t}}{\Lambda_t}\left[(r_{f,t} - \rho - \delta_a)\mathrm{d}t + \sigma_{a,t}\mathrm{d}W_t\right]\xi_{i,t+\Delta t}\right].\end{aligned} \tag{ON.88}$$

(7.2) Calculate $\widehat{\sigma}_\xi(z_{i,t}, E_t, \Gamma_t)$ as follows

$$\widehat{\sigma}_\xi(z_{i,t}, E_t, \Gamma_t) = \frac{\underline{\xi}^+_{t+\Delta t} - \underline{\xi}^-_{t+\Delta t}}{2\xi(z_{i,t}, E_t, \Gamma_t)\sqrt{\Delta t}}, \tag{ON.89}$$

where

$$\underline{\xi}^+_{t+\Delta t} = \mathbb{E}_t\left[\xi(z_{i,t+\Delta t}, E^+_{t+\Delta t}, \Gamma^+_{t+\Delta t})\right], \tag{ON.90}$$

$$\underline{\xi}^-_{t+\Delta t} = \mathbb{E}_t\left[\xi(z_{i,t+\Delta t}, E^-_{t+\Delta t}, \Gamma^-_{t+\Delta t})\right]. \tag{ON.91}$$

The expectation is taken with respect to idiosyncratic shocks in $z_{i,t+\Delta t}$.

(7.3) Solve $\underline{z}(E_t, \Gamma_t)$ using equation (ON.54), and then find the value of $\widehat{\sigma}_\xi(\underline{z}_t, E_t, \Gamma_t)$.

(7.4) If $\max\left|\widehat{\sigma}_\xi(\underline{z}_t, E_t, \Gamma_t) - \sigma_\xi(\underline{z}_t, E_t, \Gamma_t)\right| < 10^{-9}$, stop. Otherwise, jump to step (3) using $\widehat{\sigma}_\xi(\underline{z}_t, E_t, \Gamma_t)$ as the initial guess for $\sigma_\xi(\underline{z}_t, E_t, \Gamma_t)$.

(8) Solve equation (ON.61) to obtain $\widehat{q}(E_t, \Gamma_t)$.

(9) If $\max|\widehat{q}(E_t, \Gamma_t) - q(E_t, \Gamma_t)| < 10^{-9}$, stop. Otherwise, jump to step (2) using $\widehat{q}(E_t, \Gamma_t)$ as the initial guess for $q(E_t, \Gamma_t)$.

### 3.1 Higher-Degree Approximation

This section provides detailed steps for implementing the numerical approximation method in Online Appendix 5.

Following Algan, Allais and Den Haan (2008), we use the following functional form to approximate the capital share distribution $\omega_t(\tilde{z})$ defined in equation (27) in the main text with $\tilde{z} = \ln z$:

$$\omega_t(\tilde{z}) \approx g_{0,t} \exp\left( g_{1,t}(\tilde{z} - m_{1,t}) + \sum_{i=2}^{n} \left[ g_{i,t}(\tilde{z} - m_{1,t})^i - m_{i,t} \right] \right), \tag{ON.92}$$

where $m_{1,t}$, ..., $m_{n,t}$ correspond to the 1st, ..., $n$th moments of $\omega_t(\tilde{z})$, given by

$$m_{1,t} = \int_{-\infty}^{\infty} \tilde{z}\omega_t(\tilde{z})\mathrm{d}\tilde{z}, \tag{ON.93}$$

$$m_{i,t} = \int_{-\infty}^{\infty} (\tilde{z} - m_{1,t})^i \omega_t(\tilde{z})\mathrm{d}\tilde{z} \ \text{ for } i = 2, ..., n. \tag{ON.94}$$

When $n = 2$, the approximation based on equation (ON.92) is similar to our parametric approximation method in equation (32) in the main text, with $m_{1,t} = -M_t\sigma^2/2$ and $m_{2,t} = \sigma^2/2$.[6]

The evolution of $m_{i,t}$ for $i = 1, 2, ..., n$ can be derived as follows. Consider a small time interval $[t, t + \Delta t)$, equation (3) in the main text implies that $\tilde{z}_{i,t+\Delta t}$ is given by

$$\tilde{z}_{i,t+\Delta t} = (1 - \theta\Delta t)\tilde{z}_{i,t} + \sigma\sqrt{\theta\Delta t}\varepsilon_{i,t}, \quad \text{with } \varepsilon_{i,t} \sim N(0, 1). \tag{ON.95}$$

Thus, conditioning on $\tilde{z}_{i,t}$ at $t$, the probability of having $\tilde{z}_{i,t+\Delta t}$ falling in a small interval $[\tilde{z}, \tilde{z} + \Delta\tilde{z}]$ at $t + \Delta t$ is given by

$$\begin{aligned} P(\tilde{z}_{i,t+\Delta t} \in [\tilde{z}, \tilde{z} + \Delta\tilde{z}] | \tilde{z}_{i,t}) =& \Phi\left(\frac{\tilde{z} + \Delta\tilde{z} - (1 - \theta\Delta t)\tilde{z}_{i,t}}{\sigma\sqrt{\theta\Delta t}}\right) - \Phi\left(\frac{\tilde{z} - (1 - \theta\Delta t)\tilde{z}_{i,t}}{\sigma\sqrt{\theta\Delta t}}\right) \\ =& \phi\left(\frac{\tilde{z} - (1 - \theta\Delta t)\tilde{z}_{i,t}}{\sigma\sqrt{\theta\Delta t}}\right) \frac{\Delta\tilde{z}}{\sigma\sqrt{\theta\Delta t}}. \end{aligned} \tag{ON.96}$$

Equations (2) and (21) in the main text imply that $a_{i,t+\Delta t}$ is given by

$$\frac{a_{i,t+\Delta t}}{a_{i,t}} = [1 + (r_{f,t} - \delta_a - \rho)\Delta t] + \sigma_{a,t}\Delta W_t + (1 + \lambda)\left[(\kappa_t z_{i,t} - \delta_k - r_{f,t})\Delta t - \sigma_k \Delta W_t)\right] \mathbb{1}_{z_{i,t} \geq \underline{z}_t}. \tag{ON.97}$$

Thus, conditioning on $\tilde{z}_{i,t}$ at $t$ and given the aggregate shock $\Delta W_t$, to have $a_{i,t+\Delta t} \in [a, a + \Delta a]$, we

[6] Even when $n = 2$, the numerical approximation method does not produce identical results as our parametric approximation method (see Table IA.III in Online Appendix 5.3). This is because the two approximation methods subtly differ in the way they approximate the evolution of $\omega_t(\tilde{z})$. In our parametric approximation method, our assumption is that $\tilde{a}_{i,t}$ in the cross-section of firms follows a normal distribution for all $t \geq 0$, and thus $\omega_t(\tilde{z})$ follows a normal-density function for all $t \geq 0$. In particular, both $\omega_t(\tilde{z})$ and $\omega_{t+\Delta t}(\tilde{z})$ follow a normal-density function, based on which we derive a closed-form equation for the evolution of $M_t$ (see equation (38) in the main text). Then, we compute the first and second moments of $\omega_{t+\Delta t}(\tilde{z})$, which are $m_{1,t+\Delta t} = -M_{t+\Delta t}\sigma^2/2$ and $m_{2,t+\Delta t} \equiv \sigma^2/2$, respectively, using closed-form solutions. In other words, when implementing the parametric approximation method, we essentially first impose the assumption of normal-density function at both $t$ and $t + \Delta t$, then directly derive the first and second moments $m_{1,t+\Delta t}$ and $m_{2,t+\Delta t}$. By contrast, in the numerical approximation method with $n = 2$, we fit $\omega_t(\tilde{z})$ at $t$ using a normal density function as specified by equation (ON.92), and then we compute the non-parametric distribution of $\omega_{t+\Delta t}(\tilde{z})$ at $t + \Delta t$ based on the evolution of $\tilde{z}_{i,t}$ and $\tilde{a}_{i,t}$. Next, we fit $\omega_{t+\Delta t}(\tilde{z})$ using a normal density function by matching the first and second moments, $m_{1,t+\Delta t}$ and $m_{2,t+\Delta t}$, implied by $\omega_{t+\Delta t}(\tilde{z})$. In other words, when implementing the numerical approximation method, the first and second moments $m_{1,t+\Delta t}$ and $m_{2,t+\Delta t}$ are computed ex-post, after we obtain the non-parametric distribution of $\omega_{t+\Delta t}(\tilde{z})$, so the values of $m_{1,t+\Delta t}$ and $m_{2,t+\Delta t}$ are not derived based on the ex-ante assumption that $\omega_{t+\Delta t}(\tilde{z})$ follows a normal-density function. This subtle difference in the treatment of $\omega_{t+\Delta t}(\tilde{z})$ makes the evolutions of $m_{1,t+\Delta t}$ and $m_{2,t+\Delta t}$ differ slightly, making the results from the parametric approximation method and the numerical approximation method with $n = 2$ slightly different (see Table IA.III in Online Appendix 5.3).

need

$$a_{i,t} \in \left[\frac{a}{\Psi_t(\tilde{z}_{i,t})}, \frac{a+\Delta a}{\Psi_t(\tilde{z}_{i,t})}\right], \tag{ON.98}$$

where

$$\Psi_t(\tilde{z}_{i,t}) = [1 + (r_{f,t} - \delta_a - \rho)\Delta t] + \sigma_{a,t}\Delta W_t + (1+\lambda)\left[(\kappa_t z_{i,t} - \delta_k - r_{f,t})\Delta t - \sigma_k \Delta W_t)\right] \mathbb{1}_{z_{i,t} \geq \underline{z}_t}. \tag{ON.99}$$

Thus, the density $\varphi_{t+\Delta t}(a, \tilde{z})$ is given by

$$\varphi_{t+\Delta t}(a,\tilde{z})\Delta a \Delta \tilde{z} = \int_{-\infty}^{\infty} \varphi_t(a/\Psi_t(\tilde{z}_{i,t}), \tilde{z}_{i,t}) \frac{\Delta a}{\Psi_t(\tilde{z}_{i,t})} P(\tilde{z}_{i,t+\Delta t} \in [\tilde{z}, \tilde{z}+\Delta\tilde{z}] | \tilde{z}_{i,t}) \mathrm{d}\tilde{z}_{i,t}. \tag{ON.100}$$

Substituting equation (ON.96) into (ON.100), we obtain

$$\varphi_{t+\Delta t}(a,\tilde{z}) = \frac{1}{\sigma\sqrt{\theta\Delta t}} \int_{-\infty}^{\infty} \frac{1}{\Psi_t(\tilde{x})} \varphi_t(a/\Psi_t(\tilde{x}), \tilde{x}) \phi\left(\frac{\tilde{z} - (1-\theta\Delta t)\tilde{x}}{\sigma\sqrt{\theta\Delta t}}\right) \mathrm{d}\tilde{x}. \tag{ON.101}$$

By definition of equation (27) in the main text, the capital share at $t+\Delta t$ is

$$\omega_{t+\Delta t}(\tilde{z}) = \frac{1}{A_{t+\Delta t}} \int_0^{\infty} a\varphi_{t+\Delta t}(a,\tilde{z})\mathrm{d}a. \tag{ON.102}$$

Substituting equation (ON.101) into (ON.102), we obtain

$$\omega_{t+\Delta t}(\tilde{z}) = \frac{1}{\sigma\sqrt{\theta\Delta t}} \frac{1}{A_{t+\Delta t}} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \frac{a}{\Psi_t(\tilde{x})} \varphi_t(a/\Psi_t(\tilde{x}), \tilde{x})\mathrm{d}a\right) \phi\left(\frac{\tilde{z} - (1-\theta\Delta t)\tilde{x}}{\sigma\sqrt{\theta\Delta t}}\right) \mathrm{d}\tilde{x}. \tag{ON.103}$$

Define $a' = a/\Psi_t(\tilde{x})$. Using the definition in (27) in the main text, the term $\int_0^{\infty} \frac{a}{\Psi_t(\tilde{x})} \varphi_t(a/\Psi_t(\tilde{x}), \tilde{x})\mathrm{d}a$ in equation (ON.103) can be written as

$$\int_0^{\infty} \frac{a}{\Psi_t(\tilde{x})} \varphi_t(a/\Psi_t(\tilde{x}), \tilde{x})\mathrm{d}a = \Psi_t(\tilde{x}) \int_0^{\infty} a'\varphi_t(a', \tilde{x})\mathrm{d}a' = \Psi_t(\tilde{x})\omega_t(\tilde{x})A_t. \tag{ON.104}$$

Substituting equation (ON.104) into (ON.103), we obtain

$$\omega_{t+\Delta t}(\tilde{z}) = \frac{1}{\sigma\sqrt{\theta\Delta t}} \frac{A_t}{A_{t+\Delta t}} \int_{-\infty}^{\infty} \Psi_t(\tilde{x})\omega_t(\tilde{x}) \phi\left(\frac{\tilde{z} - (1-\theta\Delta t)\tilde{x}}{\sigma\sqrt{\theta\Delta t}}\right) \mathrm{d}\tilde{x}, \tag{ON.105}$$

where $\Psi_t(\tilde{x})$ is defined in equation (ON.99) with $\tilde{x} = \ln x$.

Using $\omega_{t+\Delta t}(\tilde{z})$ in equation (ON.105), we can compute the moments at $t+\Delta t$ as follows

$$m_{1,t+\Delta t} = \int_{-\infty}^{\infty} \tilde{z}\omega_{t+\Delta t}(\tilde{z})\mathrm{d}\tilde{z}, \tag{ON.106}$$

$$m_{i,t+\Delta t} = \int_{-\infty}^{\infty} (\tilde{z} - m_{1,t+\Delta t})^i \omega_{t+\Delta t}(\tilde{z})\mathrm{d}\tilde{z} \;\text{ for } i = 2, ..., n, \tag{ON.107}$$

which can be numerically integrated using Gauss-Legendre quadratures.

**Implementation Details.** Equation (ON.105) cannot be directly computed if we use a local perturbation approach because the function $\Psi_t(\tilde{x})$ has a kink at $\tilde{x} = \underline{z}_t$. Substituting out $\Psi_t(\tilde{x})$ using (ON.99), we rewrite equation (ON.105) as follows:

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{1}{\sigma\sqrt{\theta\Delta t}}\frac{A_t}{A_{t+\Delta t}}\left[\int_{-\infty}^{\widetilde{z}_t}\Psi_{l,t}\omega_t(\widetilde{x})\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{x}}{\sigma\sqrt{\theta\Delta t}}\right)\mathrm{d}\widetilde{x} + \int_{\widetilde{z}_t}^{\infty}\Psi_{h,t}(\widetilde{x})\omega_t(\widetilde{x})\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{x}}{\sigma\sqrt{\theta\Delta t}}\right)\mathrm{d}\widetilde{x}\right], \tag{ON.108}$$

where

$$\Psi_{l,t} = [1+(r_{f,t}-\delta_a-\rho)\Delta t] + \sigma_{a,t}\Delta W_t, \tag{ON.109}$$

$$\Psi_{h,t}(\widetilde{x}) = [1+(r_{f,t}-\delta_a-\rho)\Delta t] + \sigma_{a,t}\Delta W_t + (1+\lambda)\left[(\kappa_t\exp(\widetilde{x})-\delta_k-r_{f,t})\Delta t - \sigma_k\Delta W_t)\right]. \tag{ON.110}$$

By doing a change of variables, equation (ON.108) can be rewritten as

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{1}{\sigma\sqrt{\theta\Delta t}}\frac{A_t}{A_{t+\Delta t}}\left[\int_0^{\infty}\Psi_{l,t}\omega_t(\widetilde{z}_t-\widetilde{x})\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)(\widetilde{z}_t-\widetilde{x})}{\sigma\sqrt{\theta\Delta t}}\right)\mathrm{d}\widetilde{x} + \int_0^{\infty}\Psi_{h,t}(\widetilde{z}_t+\widetilde{x})\omega_t(\widetilde{z}_t+\widetilde{x})\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)(\widetilde{z}_t+\widetilde{x})}{\sigma\sqrt{\theta\Delta t}}\right)\mathrm{d}\widetilde{x}\right], \tag{ON.111}$$

so that the integration regions are $[0,\infty)$. Further, we make another change of variables by defining $\widetilde{e} = \frac{(1-\theta\Delta t)\widetilde{x}}{\sigma\sqrt{2\theta\Delta t}}$. Equation (ON.111) becomes

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{\sqrt{2}}{1-\theta\Delta t}\frac{A_t}{A_{t+\Delta t}}\left[\int_0^{\infty}\Psi_{l,t}\omega_t\left(\widetilde{z}_t-\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t}{\sigma\sqrt{\theta\Delta t}}+\sqrt{2}\widetilde{e}\right)\mathrm{d}\widetilde{e} + \int_0^{\infty}\Psi_{h,t}\left(\widetilde{z}_t+\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\omega_t\left(\widetilde{z}_t+\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\phi\left(\frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t}{\sigma\sqrt{\theta\Delta t}}-\sqrt{2}\widetilde{e}\right)\mathrm{d}\widetilde{e}\right]. \tag{ON.112}$$

Substituting the PDF of the standard normal distribution for $\phi(\cdot)$, equation (ON.112) becomes

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{1}{\sqrt{\pi}(1-\theta\Delta t)}\exp\left(-\frac{1}{2}\left[\frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t}{\sigma\sqrt{\theta\Delta t}}\right]^2\right)\frac{A_t}{A_{t+\Delta t}}\left[\int_0^{\infty}f_{l,t}(\widetilde{e}|\widetilde{z})\exp(-\widetilde{e}^2)\mathrm{d}\widetilde{e} + \int_0^{\infty}f_{h,t}(\widetilde{e}|\widetilde{z})\exp(-\widetilde{e}^2)\mathrm{d}\widetilde{e}\right], \tag{ON.113}$$

where

$$f_{l,t}(\widetilde{e}|\widetilde{z}) = \Psi_{l,t}\omega_t\left(\widetilde{z}_t-\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\exp\left(-\frac{\sqrt{2}(\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t)}{\sigma\sqrt{\theta\Delta t}}\widetilde{e}\right), \tag{ON.114}$$

$$f_{h,t}(\widetilde{e}|\widetilde{z}) = \Psi_{h,t}\left(\widetilde{z}_t+\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\omega_t\left(\widetilde{z}_t+\frac{\sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right)\exp\left(\frac{\sqrt{2}(\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t)}{\sigma\sqrt{\theta\Delta t}}\widetilde{e}\right). \tag{ON.115}$$

Equation (ON.113) can be computed using one-sided Gauss-Hermite quadrature (Steen, Byrne and Gelbard, 1969).[7] The results presented in Table IA.III in Online Appendix 5.3 are computed

[7] The term $\exp\left(\frac{\sqrt{2}(\widetilde{z}-(1-\theta\Delta t)\widetilde{z}_t)}{\sigma\sqrt{\theta\Delta t}}\widetilde{e}\right)$ can introduce large numerical errors if it is too large. Thus, the choice of $\Delta t$ cannot be too small.

based on this integration method.

Alternatively, we can compute an approximation of equation (ON.105) by changing the timing assumption of our model. Note that equation (ON.105) is obtained based on the timing assumption that the shock to idiosyncratic productivity $z_{i,t+\Delta t}$ at $t+\Delta t$ occurs after capital accumulation over $[t, t+\Delta t)$ based on productivity $z_{i,t}$. When $\Delta t \approx 0$, this timing assumption yields similar results to an alternative timing assumption under which the value of idiosyncratic productivity $z_{i,t+\Delta t}$ at $t+\Delta t$ is realized at the beginning of $[t, t+\Delta t)$. Then capital accumulation over $[t, t+\Delta t)$ is based on $z_{i,t+\Delta t}$. In this case, equation (ON.105) becomes

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{\Psi_t(\widetilde{z})}{\sigma\sqrt{\theta\Delta t}} \frac{A_t}{A_{t+\Delta t}} \int_{-\infty}^{\infty} \omega_t(\widetilde{x}) \phi\left(\frac{\widetilde{z} - (1-\theta\Delta t)\widetilde{x}}{\sigma\sqrt{\theta\Delta t}}\right) \mathrm{d}\widetilde{x}, \tag{ON.116}$$

which can be rewritten as (by making a change of variable $\widetilde{e} = \frac{\widetilde{z}-(1-\theta\Delta t)\widetilde{x}}{\sigma\sqrt{2\theta\Delta t}}$):

$$\omega_{t+\Delta t}(\widetilde{z}) = \frac{\Psi_t(\widetilde{z})}{\sqrt{\pi}(1-\theta\Delta t)} \frac{A_t}{A_{t+\Delta t}} \int_{-\infty}^{\infty} \omega_t\left(\frac{\widetilde{z} - \sigma\sqrt{2\theta\Delta t}\widetilde{e}}{1-\theta\Delta t}\right) \exp(-\widetilde{e}^2)\mathrm{d}\widetilde{e}. \tag{ON.117}$$

Equation (ON.117) can be easily implemented in dynare because the kinked function $\Psi_t(\widetilde{z})$ is not part of the integrand. Thus, the integral in equation (ON.117) can be easily computed using Gauss-Hermite quadratures. The integral in equations (ON.106) and (ON.107) can then be computed as the sum of two integrals over $[-\infty, \underline{\widetilde{z}})$ and $[\underline{\widetilde{z}}, +\infty)$, respectively. The integral in each interval can be computed using Gauss-Legendre quadratures.

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